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CALCULUS

Early Transcendentals

an Open Text

Adapted for

ATHABASCA UNIVERSITY

MATH 266 – CALCULUS II

NOVEMBER 2017 EDITION

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by Lyryx Learning
based on the original text by D. Guichard

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Base Text Revision History
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Extensive edits, additions, and revisions have been completed by the editorial team at Lyryx Learning.
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Adapted for Athabasca University, October 2017

- G. Hartman:
  - Many new exercises are included, adapted by Lyryx from APEX Calculus. The following exercises are from APEX: 1.2.2 to 1.2.21, 3.1.15 to 3.1.56, 3.2.13 to 3.2.35, 3.3.11 to 3.3.22, 3.4.11 to 3.4.33, 3.7.11 to 3.7.37, 4.2.10 to 4.2.26
  - New content on Hyperbolic Functions 1.7 and Inverse Hyperbolic Functions 2.2 is included. These sections were adapted by Lyryx from the section "Hyperbolic Functions" in APEX Calculus.
  - APEX Calculus: Version 3.0 written by G. Hartman. T. Siemers and D. Chalishajar of the Virginia Military Institiue and B. Heinold of Mount Saint Mary's University also contributed to APEX Calculus. This material is released under Creative Commons license CC BY-NC (https://creativecommons.org/licenses/by-nc/4.0/). See http://www.apexcalculus.com/ for more information and original version.
- OpenStax: New content is included; Table of Integrals in Additional Material. This section was adapted by Lyryx from the section of the same name in Calculus Volume 1 by OpenStax. This material is released under Creative Commons license CC BY-NC-SA (https://creativecommons.org/licenses/by-nc-sa/4.0/). Download for free at (http://cnx.org/contents/8b89d172-2927-466f-8661-01abc7ccdba4@2.66).

2017 A

- Lyryx:
  - Front matter has been updated including cover page, copyright, and revision pages.
  - Several examples and exercises from Chapter 15 and 16 have been rewritten or removed.
  - Order and name of topics in Chapter 15 and Chapter 16 have been revised.

2016 B

- D. Guichard: New content developed for the Three Dimensions, Vector Functions, and Vector Calculus chapters.
- Lyryx: Exercise numbering has been updated to restart with each section.
- G. Hartman: New content on Riemann Sums 3.5 is included. This section was adapted by Lyryx from the section of the same name in APEX Calculus 3.0. See above for APEX Calculus license information.

Continued on next page ...

2016 A	• Lyryx: The layout and appearance of the text has been updated, including the title page and newly designed back cover.
2015 A	• J. Ling: Addition of new exercises and proofs throughout. Revised arrangement of topics in the Application of Derivatives chapter. Continuity section revised to include additional explanations of content and additional examples.
2014 A	• M. Cavers: Addition of new material and images particularly in the Review chapter. • M. Blenkinsop: Addition of content including Linear and Higher Order Approximations section.
2012 A	• **Original text** by D. Guichard of Whitman College, the single variable material is a modification and expansion of notes written and released by N. Koblitz of the University of Washington. That version also contains exercises and examples from *Elementary Calculus: An Approach Using Infinitesimals*, written by H. J. Keisler of the University of Wisconsin under a Creative Commons license (see `http://www.math.wisc.edu/~keisler/calc.html`). A. Schueller, B. Balof, and M. Wills all of Whitman College, have also contributed content. This material is released under the Creative Commons Attribution-NonCommercial-ShareAlike License (`http://creativecommons.org/licenses/by-nc-sa/3.0/`). See `http://communitycalculus.org` for more information.

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Introduction and Review

The emphasis in this course is on problems—doing calculations and story problems. To master problem solving one needs a tremendous amount of practice doing problems. The more problems you do the better you will be at doing them, as patterns will start to emerge in both the problems and in successful approaches to them. You will learn quickly and effectively if you devote some time to doing problems every day.

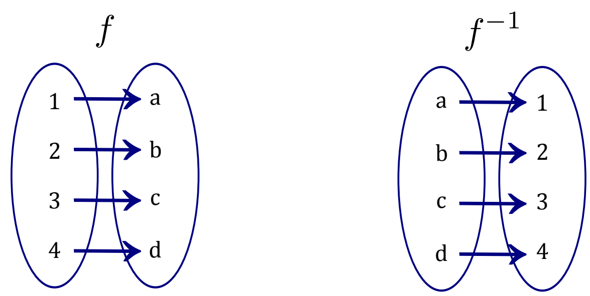
Typically the most difficult problems are story problems, since they require some effort before you can begin calculating. Here are some pointers for doing story problems:

1. Carefully read each problem twice before writing anything.
2. Assign letters to quantities that are described only in words; draw a diagram if appropriate.
3. Decide which letters are constants and which are variables. A letter stands for a constant if its value remains the same throughout the problem.
4. Using mathematical notation, write down what you know and then write down what you want to find.
5. Decide what category of problem it is (this might be obvious if the problem comes at the end of a particular chapter, but will not necessarily be so obvious if it comes on an exam covering several chapters).
6. Double check each step as you go along; don't wait until the end to check your work.
7. Use common sense; if an answer is out of the range of practical possibilities, then check your work to see where you went wrong.

Unit 1: Inverse Functions

1.1 Inverse Functions

In mathematics, an *inverse* is a function that serves to "undo" another function. That is, if $f(x)$ produces y, then putting y into the inverse of f produces the output x. A function f that has an inverse is called invertible and the inverse is denoted by f^{-1}. It is best to illustrate inverses using an arrow diagram:



Notice how f maps 1 to a, and f^{-1} undoes this, that is, f^{-1} maps a back to 1. Don't confuse $f^{-1}(x)$ with exponentiation: the inverse f^{-1} is *different* from $\frac{1}{f(x)}$.

Not every function has an inverse. It is easy to see that if a function $f(x)$ is going to have an inverse, then $f(x)$ *never* takes on the same value twice. We give this property a special name.

A function $f(x)$ is called **one-to-one** if every element of the range corresponds to *exactly* one element of the domain. Similar to the Vertical Line Test (VLT) for functions, we have the Horizontal Line Test (HLT) for the one-to-one property.

Theorem 1.1: The Horizontal Line Test

A function is one-to-one if and only if there is no horizontal line that intersects its graph more than once.

Example 1.2: Parabola is Not One-to-one

The parabola $f(x) = x^2$ it not one-to-one because it does not satisfy the horizontal line test. For example, the horizontal line $y = 1$ intersects the parabola at two points, when $x = -1$ and $x = 1$.

We now formally define the inverse of a function.

Definition 1.3: Inverse of a Function

Let $f(x)$ and $g(x)$ be two one-to-one functions. If $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$ then we say that $f(x)$ and $g(x)$ are **inverses** *of each other. We denote $g(x)$ (the inverse of $f(x)$) by $g(x) = f^{-1}(x)$.*

Thus, if f maps x to y, then f^{-1} maps y back to x. This gives rise to the *cancellation formulas*:

$$f^{-1}(f(x)) = x, \quad \text{for every } x \text{ in the domain of } f(x),$$

$$f(f^{-1}(x)) = x, \quad \text{for every } x \text{ in the domain of } f^{-1}(x).$$

Example 1.4: Finding the Inverse at Specific Values

If $f(x) = x^9 + 2x^7 + x + 1$, find $f^{-1}(5)$ and $f^{-1}(1)$.

Solution. Rather than trying to compute a formula for f^{-1} and then computing $f^{-1}(5)$, we can simply find a number c such that f evaluated at c gives 5. Note that subbing in some simple values ($x = -3, -2, 1, 0, 1, 2, 3$) and evaluating $f(x)$ we eventually find that $f(1) = 1^9 + 2(1^7) + 1 + 1 = 5$ and $f(0) = 1$. Therefore, $f^{-1}(5) = 1$ and $f^{-1}(1) = 0$. ♣

To compute the equation of the inverse of a function we use the following *guidelines*.

Guidelines for Computing Inverses

1. Write down $y = f(x)$.
2. Solve for x in terms of y.
3. Switch the x's and y's.
4. The result is $y = f^{-1}(x)$.

Example 1.5: Finding the Inverse Function

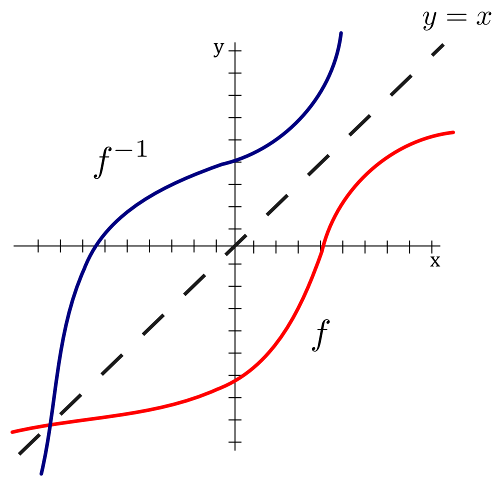
We find the inverse of the function $f(x) = 2x^3 + 1$.

Solution. Starting with $y = 2x^3 + 1$ we solve for x as follows:

$$y - 1 = 2x^3 \quad \to \quad \frac{y-1}{2} = x^3 \quad \to \quad x = \sqrt[3]{\frac{y-1}{2}}.$$

Therefore, $f^{-1}(x) = \sqrt[3]{\frac{x-1}{2}}$. ♣

This example shows how to find the inverse of a function *algebraically*. But what about finding the inverse of a function *graphically*? Step 3 (switching x and y) gives us a good graphical technique to find the inverse, namely, for each point (a, b) where $f(a) = b$, sketch the point (b, a) for the inverse. More formally, to obtain $f^{-1}(x)$ *reflect* the graph $f(x)$ about the line $y = x$.



Exercises for 1.1

Exercise 1.1.1 *Is the function $f(x) = |x|$ one-to-one?*

Exercise 1.1.2 *Find a formula for the inverse of the function* $f(x) = \dfrac{x+2}{x-2}$.

1.2 Derivatives of Inverse Functions

Suppose we wanted to find the *derivative of the inverse*, but do not have an actual formula for the inverse function? Then we can use the following derivative formula for the inverse evaluated at a.

Derivative of $f^{-1}(a)$

Given an invertible function $f(x)$, the derivative of its inverse function $f^{-1}(x)$ evaluated at $x = a$ is:

$$\left[f^{-1}\right]'(a) = \frac{1}{f'\left[f^{-1}(a)\right]}$$

To see why this is true, start with the function $y = f^{-1}(x)$. Write this as $x = f(y)$ and differentiate both sides implicitly with respect to x using the chain rule:

$$1 = f'(y) \cdot \frac{dy}{dx}.$$

Thus,

$$\frac{dy}{dx} = \frac{1}{f'(y)},$$

but $y = f^{-1}(x)$, thus,

$$\left[f^{-1}\right]'(x) = \frac{1}{f'\left[f^{-1}(x)\right]}.$$

At the point $x = a$ this becomes:

$$\left[f^{-1}\right]'(a) = \frac{1}{f'\left[f^{-1}(a)\right]}$$

Example 1.6: Derivatives of Inverse Functions

Suppose $f(x) = x^5 + 2x^3 + 7x + 1$. *Find* $\left[f^{-1}\right]'(1)$.

Solution. First we should show that f^{-1} exists (i.e. that f is one-to-one). In this case the derivative $f'(x) = 5x^4 + 6x^2 + 7$ is strictly greater than 0 for all x, so f is strictly increasing and thus one-to-one.

It's difficult to find the inverse of $f(x)$ (and then take the derivative). Thus, we use the above formula evaluated at 1:

$$\left[f^{-1}\right]'(1) = \frac{1}{f'\left[f^{-1}(1)\right]}.$$

Note that to use this formula we need to know what $f^{-1}(1)$ is, and the derivative $f'(x)$. To find $f^{-1}(1)$ we make a table of values (plugging in $x = -3, -2, -1, 0, 1, 2, 3$ into $f(x)$) and see what value of x gives 1. We omit the table and simply observe that $f(0) = 1$. Thus,

$$f^{-1}(1) = 0.$$

Now we have:

$$\left[f^{-1}\right]'(1) = \frac{1}{f'(0)}.$$

And so, $f'(0) = 7$. Therefore,

$$\left[f^{-1}\right]'(1) = \frac{1}{7}.$$

♣

Exercises for 1.2

Exercise 1.2.1 *Suppose* $f(x) = x^3 + 4x + 2$. *Find the slope of the tangent line to the graph of* $g(x) = xf^{-1}(x)$ *at the point where* $x = 7$.

In the following, verify that the given functions are inverses.

Exercise 1.2.2 $f(x) = 2x + 6$ *and* $g(x) = \frac{1}{2}x - 3$

Exercise 1.2.3 $f(x) = x^2 + 6x + 11$, $x \geq 3$ *and*
$g(x) = \sqrt{x-2} - 3$, $x \geq 2$

Exercise 1.2.4 $f(x) = \dfrac{3}{x-5}$, $x \neq 5$ *and*
$g(x) = \dfrac{3+5x}{x}$, $x \neq 0$

Exercise 1.2.5 $f(x) = \dfrac{x+1}{x-1}$, $x \neq 1$ *and* $g(x) = f(x)$

In the following, an invertible function $f(x)$ is given along with a point that lies on its graph. Evaluate $\left(f^{-1}\right)'(x)$ at the indicated value.

Exercise 1.2.6 $f(x) = 5x+10$
Point$= (2,20)$
Evaluate $\left(f^{-1}\right)'(20)$

Exercise 1.2.7 $f(x) = x^2-2x+4$, $x \geq 1$
Point$= (3,7)$
Evaluate $\left(f^{-1}\right)'(7)$

Exercise 1.2.8 $f(x) = \sin 2x$, $-\pi/4 \leq x \leq \pi/4$
Point$= (\pi/6, \sqrt{3}/2)$
Evaluate $\left(f^{-1}\right)'(\sqrt{3}/2)$

Exercise 1.2.9 $f(x) = x^3-6x^2+15x-2$
Point$= (1,8)$
Evaluate $\left(f^{-1}\right)'(8)$

Exercise 1.2.10 $f(x) = \dfrac{1}{1+x^2}$, $x \geq 0$
Point$= (1,1/2)$
Evaluate $\left(f^{-1}\right)'(1/2)$

Exercise 1.2.11 $f(x) = 6e^{3x}$
Point$= (0,6)$
Evaluate $\left(f^{-1}\right)'(6)$

In the following, compute the derivative of the given function.

Exercise 1.2.12 $h(t) = \sin^{-1}(2t)$

Exercise 1.2.13 $f(t) = \sec^{-1}(2t)$

Exercise 1.2.14 $g(x) = \tan^{-1}(2x)$

Exercise 1.2.15 $f(x) = x\sin^{-1}x$

Exercise 1.2.16 $g(t) = \sin t \cos^{-1} t$

Exercise 1.2.17 $f(t) = \ln t e^t$

Exercise 1.2.18 $h(x) = \dfrac{\sin^{-1}x}{\cos^{-1}x}$

Exercise 1.2.19 $g(x) = \tan^{-1}(\sqrt{x})$

Exercise 1.2.20 $f(x) = \sec^{-1}(1/x)$

Exercise 1.2.21 $f(x) = \sin(\sin^{-1}x)$

1.3 Exponential Functions

An **exponential function** is a function of the form $f(x) = a^x$, where a is a constant. Examples are 2^x, 10^x and $(1/2)^x$. To more formally define the exponential function we look at various kinds of input values.

It is obvious that $a^5 = a \cdot a \cdot a \cdot a \cdot a$ and $a^3 = a \cdot a \cdot a$, but when we consider an exponential function a^x we can't be limited to substituting integers for x. What does $a^{2.5}$ or $a^{-1.3}$ or a^{π} mean? And is it really true that $a^{2.5}a^{-1.3} = a^{2.5-1.3}$? The answer to the first question is actually quite difficult, so we will evade it; the answer to the second question is "yes."

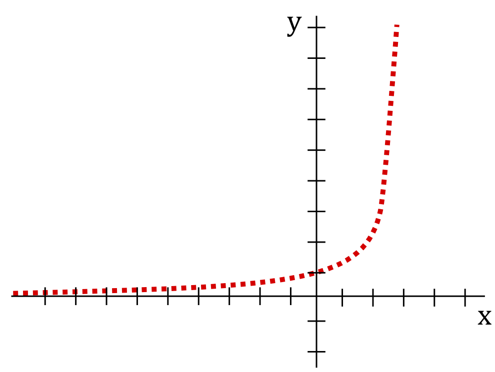
We'll evade the full answer to the hard question, but we have to know something about exponential functions. You need first to understand that since it's not "obvious" what 2^x should mean, we are really free to make it mean whatever we want, so long as we keep the behavior that *is* obvious, namely, when x is a positive integer. What else do we want to be true about 2^x? We want the properties of the previous two paragraphs to be true for all exponents: $2^x 2^y = 2^{x+y}$ and $(2^x)^y = 2^{xy}$.

After the positive integers, the next easiest number to understand is 0: $2^0 = 1$. You have presumably learned this fact in the past; why is it true? It is true precisely because we want $2^a 2^b = 2^{a+b}$ to be true about the function 2^x. We need it to be true that $2^0 2^x = 2^{0+x} = 2^x$, and this only works if $2^0 = 1$. The same argument implies that $a^0 = 1$ for any a.

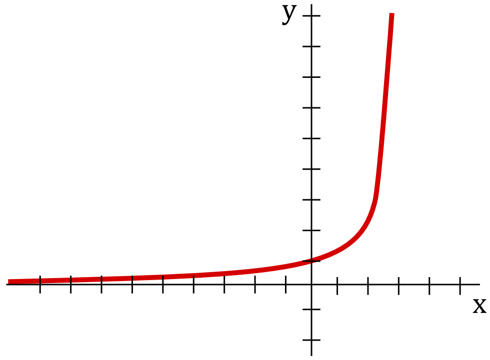
The next easiest set of numbers to understand is the negative integers: for example, $2^{-3} = 1/2^3$. We know that whatever 2^{-3} means it must be that $2^{-3}2^3 = 2^{-3+3} = 2^0 = 1$, which means that 2^{-3} must be $1/2^3$. In fact, by the same argument, once we know what 2^x means for some value of x, 2^{-x} must be $1/2^x$ and more generally $a^{-x} = 1/a^x$.

Next, consider an exponent $1/q$, where q is a positive integer. We want it to be true that $(2^x)^y = 2^{xy}$, so $(2^{1/q})^q = 2$. This means that $2^{1/q}$ is a q-th root of 2, $2^{1/q} = \sqrt[q]{2}$. This is all we need to understand that $2^{p/q} = (2^{1/q})^p = (\sqrt[q]{2})^p$ and $a^{p/q} = (a^{1/q})^p = (\sqrt[q]{a})^p$.

What's left is the hard part: what does 2^x mean when x cannot be written as a fraction, like $x = \sqrt{2}$ or $x = \pi$? What we know so far is how to assign meaning to 2^x whenever $x = p/q$. If we were to graph a^x (for some $a > 1$) at points $x = p/q$ then we'd see something like this:

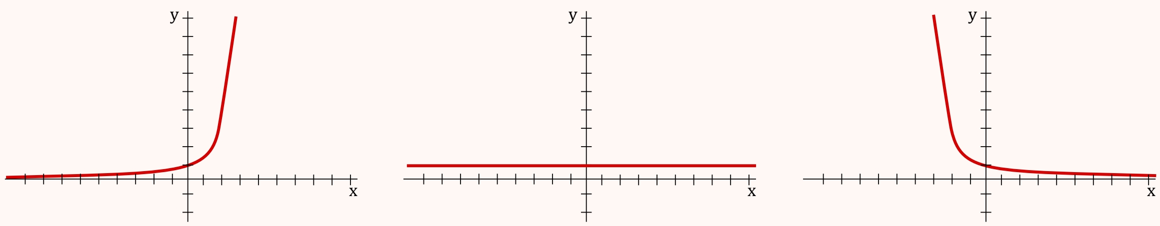


This is a poor picture, but it illustrates a series of individual points above the rational numbers on the x-axis. There are really a lot of "holes" in the curve, above $x = \pi$, for example. But (this is the hard part) it is possible to prove that the holes can be "filled in", and that the resulting function, called a^x, really does have the properties we want, namely that $a^x a^y = a^{x+y}$ and $(a^x)^y = a^{xy}$. Such a graph would then look like this:



Three Types of Exponential Functions

There are *three kinds* of exponential functions $f(x) = a^x$ depending on whether $a > 1$, $a = 1$ or $0 < a < 1$:



$f(x) = a^x$, $a > 1$ | $f(x) = 1^x$ | $f(x) = a^x$, $0 < a < 1$

Properties of Exponential Functions

The first thing to note is that if $a < 0$ then problems can occur. Observe that if $a = -1$ then $(-1)^x$ is not defined for every x. For example, $x = 1/2$ is a square root and gives $(-1)^{1/2} = \sqrt{-1}$ which is not a real number.

Exponential Function Properties

- *Only defined for positive a:* a^x is only defined for all real x if $a > 0$
- *Always positive:* $a^x > 0$, for all x
- *Exponent rules:*
  1. $a^x a^y = a^{x+y}$
  2. $\dfrac{a^x}{a^y} = a^{x-y}$
  3. $(a^x)^y = a^{xy} = a^{yx} = (a^y)^x$
  4. $a^x b^x = (ab)^x$
- *Long term behaviour:* If $a > 1$, then $a^x \to \infty$ as $x \to \infty$ and $a^x \to 0$ as $x \to -\infty$.

The last property can be observed from the graph. If $a > 1$, then as x gets larger and larger, so does a^x. On the other hand, as x gets large and negative, the function approaches the x-axis, that is, a^x approaches 0.

Example 1.7: Reflection of Exponential

Determine an equation of the function after reflecting $y = 2^x$ about the line $x = -2$.

Solution. First reflect about the y-axis to get $y = 2^{-x}$. Now shift by $2 \times 2 = 4$ units to the *left* to get $y = 2^{-(x+4)}$. Side note: Can you see why this sequence of transformations is the same as reflection in the line $x = -2$? Can you come up with a general rule for these types of reflections? ♣

Example 1.8: Determine the Exponential Function

Determine the exponential function $f(x) = ka^x$ that passes through the points $(1,6)$ and $(2,18)$.

Solution. We substitute our two points into the equation to get:

$$x = 1, y = 6 \rightarrow 6 = ka^1$$

$$x = 2, y = 18 \rightarrow 18 = ka^2$$

This gives us $6 = ka$ and $18 = ka^2$. The first equation is $k = 6/a$ and subbing this into the second gives: $18 = (6/a)a^2$. Thus, $18 = 6a$ and $a = 3$. Now we can see from $6 = ka$ that $k = 2$. Therefore, the exponential function is

$$f(x) = 2 \cdot 3^x.$$

♣

There is one base that is so important and convenient that we give it a special symbol. This number is denoted by $e = 2.71828\ldots$ (and is an irrational number). Its *importance* stems from the fact that it simplifies many formulas of Calculus and also shows up in other fields of mathematics.

Example 1.9: Domain of Function with Exponential

Find the domain of $f(x) = \dfrac{1}{\sqrt{e^x + 1}}$.

Solution. For domain, we cannot divide by zero or take the square root of negative numbers. Note that one of the properties of exponentials is that they are always positive! Thus, $e^x + 1 > 0$ (in fact, as $e^x > 0$ we actually have that $e^x + 1$ is at least one). Therefore, $e^x + 1$ is never zero nor negative, and gives no restrictions on x. Thus, the domain is $\mathbb{R}$. ♣

Exercises for 1.3

Exercise 1.3.1 *Determine an equation of the function $y = a^x$ passing through the point* $(3,8)$.

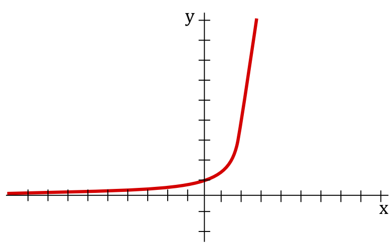
Exercise 1.3.2 *Find the y-intercept of $f(x) = 4^x + 6$.*

Exercise 1.3.3 *Find the y-intercept of $f(x) = 2\left(\frac{1}{2}\right)^x$.*

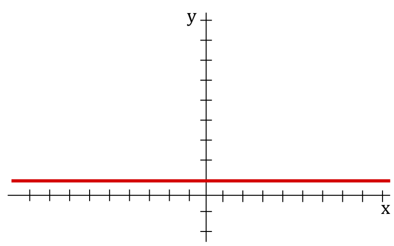
Exercise 1.3.4 *Find the domain of $y = e^{-x} + e^{\frac{1}{x}}$.*

1.4 Logarithms

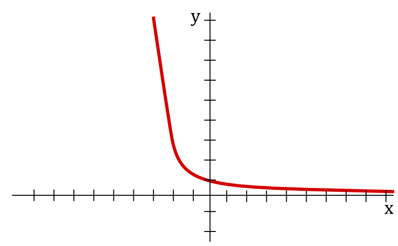
Recall the *three kinds* of exponential functions $f(x) = a^x$ depending on whether $0 < a < 1$, $a = 1$ or $a > 1$:



$f(x) = a^x$
$a > 1$



$f(x) = 1^x$



$f(x) = a^x$
$0 < a < 1$

So long as $a \neq 1$, the function $f(x) = a^x$ satisfies the horizontal line test and therefore has an inverse. We call the *inverse of* a^x the **logarithmic function with base a** and denote it by $\log_a$. In particular,

$$\log_a x = y \iff a^y = x.$$

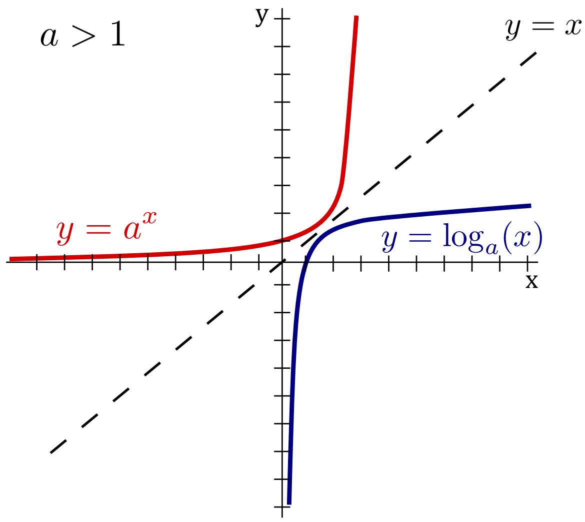
The *cancellation formulas* for logs are:

$$\log_a(a^x) = x, \quad \text{for every } x \in \mathbb{R},$$

$$a^{\log_a(x)} = x, \quad \text{for every } x > 0.$$

Since the function $f(x) = a^x$ for $a \neq 1$ has domain $\mathbb{R}$ and range $(0,\infty)$, the logarithmic function has domain $(0,\infty)$ and range $\mathbb{R}$. For the most part, we only focus on logarithms with a base larger than 1 (i.e., $a > 1$)

as these are the most important.



Notice that every logarithm passes through the point $(1,0)$ in the same way that every exponential function passes through the point $(0,1)$.

Some properties of logarithms are as follows.

Logarithm Properties

Let A, B be positive numbers and $b > 0$ $(b \neq 1)$ be a base.

- $\log_b(AB) = \log_b A + \log_b B$,
- $\log_b \left(\frac{A}{B}\right) = \log_b A - \log_b B$,
- $\log_b(A^n) = n \log_b A$, where n is any real number.

Example 1.10: Compute Lorarithms

To compute $\log_2(24) - \log_2(3)$ *we can do the following:*

$$\log_2(24) - \log_2(3) = \log_2\left(\frac{24}{3}\right) = \log_2(8) = 3,$$

since $2^3 = 8$.

The Natural Logarithm

As mentioned earlier for exponential functions, the number $e \approx 2.71828\ldots$ is the most convenient base to use in Calculus. For this reason we give the logarithm with base e a special name: **the natural logarithm**. We also give it special notation:

$$\log_e x = \ln x.$$

You may pronounce ln as either: "el - en", "lawn", or refer to it as "natural log". The above properties of logarithms also apply to the natural logarithm.

Often we need to turn a logarithm (in a different base) into a natural logarithm. This gives rise to the *change of base formula*.

Change of Base Formula

$$\log_a x = \frac{\ln x}{\ln a}.$$

Example 1.11: Combine Logarithms

Write $\ln A + 2\ln B - \ln C$ *as a single logarithm.*

Solution. Using properties of logarithms, we have,

$$\begin{aligned}\ln A + 2\ln B - \ln C &= \ln A + \ln B^2 - \ln C \\ &= \ln(AB^2) - \ln C \\ &= \ln \frac{AB^2}{C}\end{aligned}$$

♣

Example 1.12: Solve Exponential Equations using Logarithms

If $e^{x+2} = 6e^{2x}$, *then solve for* x.

Solution. Taking the natural logarithm of both sides and noting the cancellation formulas (along with $\ln e = 1$), we have:

$$\begin{aligned} e^{x+2} &= 6e^{2x} \\ \ln e^{x+2} &= \ln(6e^{2x}) \\ x+2 &= \ln 6 + \ln e^{2x} \\ x+2 &= \ln 6 + 2x \\ x &= 2 - \ln 6 \end{aligned}$$

♣

Example 1.13: Solve Logarithm Equations using Exponentials

If $\ln(2x-1) = 2\ln(x)$, *then solve for* x.

Solution. "Taking e" of both sides and noting the cancellation formulas, we have:

$$\begin{aligned} e^{\ln(2x-1)} &= e^{2\ln(x)} \\ (2x-1) &= e^{\ln(x^2)} \\ 2x-1 &= x^2 \\ x^2-2x+1 &= 0 \\ (x-1)^2 &= 0 \end{aligned}$$

Therefore, the solution is $x = 1$. ♣

Exercises for 1.4

Exercise 1.4.1 *Expand* $\log_{10}((x+45)^7(x-2))$.

Exercise 1.4.2 *Expand* $\log_2 \dfrac{x^3}{3x-5+(7/x)}$.

Exercise 1.4.3 *Write* $\log_2 3x + 17\log_2(x-2) - 2\log_2(x^2+4x+1)$ *as a single logarithm.*

Exercise 1.4.4 *Solve* $\log_2(1+\sqrt{x}) = 6$ *for* x.

Exercise 1.4.5 *Solve* $2^{x^2} = 8$ *for* x.

Exercise 1.4.6 *Solve* $\log_2(\log_3(x)) = 1$ *for* x.

1.5 Derivatives of Exponential & Logarithmic Functions

As with the sine function, we don't know anything about derivatives that allows us to compute the derivatives of the exponential and logarithmic functions without going back to basics. Let's do a little work with the definition again:

$$\begin{aligned}
\frac{d}{dx}a^x &= \lim_{\Delta x\to 0}\frac{a^{x+\Delta x}-a^x}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{a^x a^{\Delta x}-a^x}{\Delta x}\\
&= \lim_{\Delta x\to 0} a^x\frac{a^{\Delta x}-1}{\Delta x}\\
&= a^x\lim_{\Delta x\to 0}\frac{a^{\Delta x}-1}{\Delta x}
\end{aligned}$$

There are two interesting things to note here: As in the case of the sine function we are left with a limit that involves Δx but not x, which means that if $\lim\limits_{\Delta x\to 0}(a^{\Delta x}-1)/\Delta x$ exists, then it is a constant number. This means that a^x has a remarkable property: its derivative is a constant times itself.

We earlier remarked that the hardest limit we would compute is $\lim\limits_{x\to 0}\sin x/x = 1$; we now have a limit that is just a bit too hard to include here. In fact the hard part is to see that $\lim\limits_{\Delta x\to 0}(a^{\Delta x}-1)/\Delta x$ even exists—does this fraction really get closer and closer to some fixed value? Yes it does, but we will not prove this fact.

We can look at some examples. Consider $(2^x-1)/x$ for some small values of x: 1, 0.828427124, 0.756828460, 0.724061864, 0.70838051, 0.70070877 when x is 1, $1/2$, $1/4$, $1/8$, $1/16$, $1/32$, respectively. It looks like this is settling in around 0.7, which turns out to be true (but the limit is not exactly 0.7). Consider next $(3^x-1)/x$: 2, 1.464101616, 1.264296052, 1.177621520, 1.13720773, 1.11768854, at the same values of x. It turns out to be true that in the limit this is about 1.1. Two examples don't establish a pattern, but if you do more examples you will find that the limit varies directly with the value of a: bigger a, bigger limit; smaller a, smaller limit. As we can already see, some of these limits will be less than 1 and some larger than 1. Somewhere between $a=2$ and $a=3$ the limit will be exactly 1; the value at which this happens is called e, so that

$$\lim_{\Delta x\to 0}\frac{e^{\Delta x}-1}{\Delta x}=1.$$

As you might guess from our two examples, e is closer to 3 than to 2, and in fact $e\approx 2.718$.

Now we see that the function e^x has a truly remarkable property:

$$\begin{aligned}
\frac{d}{dx}e^x &= \lim_{\Delta x\to 0}\frac{e^{x+\Delta x}-e^x}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{e^x e^{\Delta x}-e^x}{\Delta x}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\Delta x \to 0} e^x \frac{e^{\Delta x} - 1}{\Delta x} \\
&= e^x \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} \\
&= e^x
\end{aligned}
$$

That is, e^x is its own derivative, or in other words the slope of e^x is the same as its height, or the same as its second coordinate: The function $f(x) = e^x$ goes through the point (z, e^z) and has slope e^z there, no matter what z is. It is sometimes convenient to express the function e^x without an exponent, since complicated exponents can be hard to read. In such cases we use $\exp(x)$, e.g., $\exp(1+x^2)$ instead of e^{1+x^2}.

What about the logarithm function? This too is hard, but as the cosine function was easier to do once the sine was done, so is the logarithm easier to do now that we know the derivative of the exponential function. Let's start with $\log_e x$, which as you probably know is often abbreviated $\ln x$ and called the "natural logarithm" function.

Consider the relationship between the two functions, namely, that they are inverses, that one "undoes" the other. Graphically this means that they have the same graph except that one is "flipped" or "reflected" through the line $y = x$:

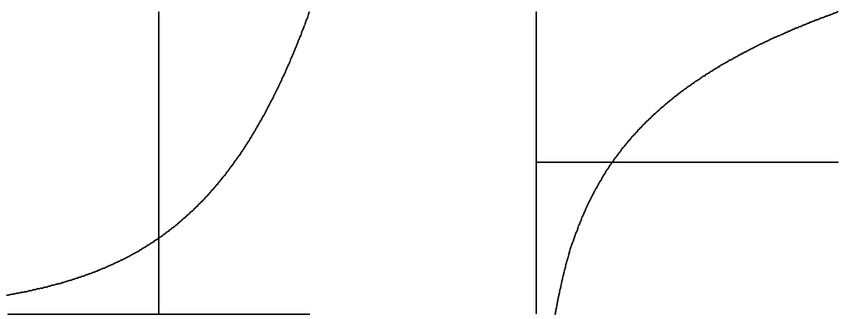


Figure 1.1: The exponential and logarithmic functions.

This means that the slopes of these two functions are closely related as well: For example, the slope of e^x is e at $x = 1$; at the corresponding point on the $\ln(x)$ curve, the slope must be $1/e$, because the "rise" and the "run" have been interchanged. Since the slope of e^x is e at the point $(1, e)$, the slope of $\ln(x)$ is $1/e$ at the point $(e, 1)$.

More generally, we know that the slope of e^x is e^z at the point (z, e^z), so the slope of $\ln(x)$ is $1/e^z$ at (e^z, z). In other words, the slope of $\ln x$ is the reciprocal of the first coordinate at any point; this means that the slope of $\ln x$ at $(x, \ln x)$ is $1/x$. The upshot is:

$$\frac{d}{dx} \ln x = \frac{1}{x}.$$

We have discussed this from the point of view of the graphs, which is easy to understand but is not normally considered a rigorous proof—it is too easy to be led astray by pictures that seem reasonable but that miss some hard point. It is possible to do this derivation without resorting to pictures, and indeed we will see an alternate approach soon.

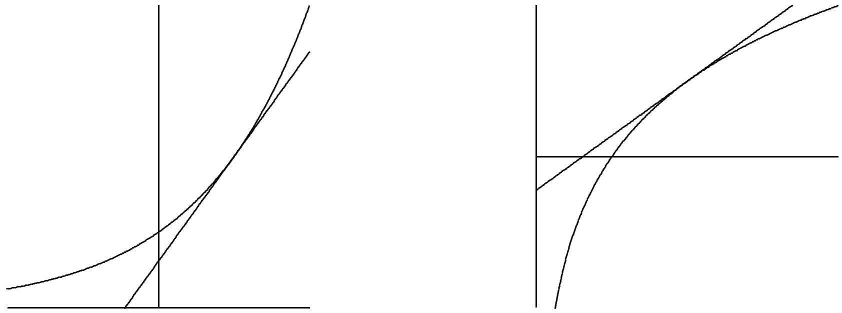


Figure 1.2: The exponential and logarithmic functions.

Note that $\ln x$ is defined only for $x > 0$. It is sometimes useful to consider the function $\ln|x|$, a function defined for $x \neq 0$. When $x < 0$, $\ln|x| = \ln(-x)$ and

$$\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}.$$

Thus whether x is positive or negative, the derivative is the same.

What about the functions a^x and $\log_a x$? We know that the derivative of a^x is some constant times a^x itself, but what constant? Remember that "the logarithm is the exponent" and you will see that $a = e^{\ln a}$. Then

$$a^x = (e^{\ln a})^x = e^{x\ln a},$$

and we can compute the derivative using the chain rule:

$$\frac{d}{dx}a^x = \frac{d}{dx}(e^{\ln a})^x = \frac{d}{dx}e^{x\ln a} = (\ln a)e^{x\ln a} = (\ln a)a^x.$$

The constant is simply $\ln a$. Likewise we can compute the derivative of the logarithm function $\log_a x$. Since

$$x = e^{\ln x}$$

we can take the logarithm base a of both sides to get

$$\log_a(x) = \log_a(e^{\ln x}) = \ln x \log_a e.$$

Then

$$\frac{d}{dx}\log_a x = \frac{1}{x}\log_a e.$$

This is a perfectly good answer, but we can improve it slightly. Since

$$\begin{aligned} a &= e^{\ln a} \\ \log_a(a) &= \log_a(e^{\ln a}) = \ln a \log_a e \\ 1 &= \ln a \log_a e \\ \frac{1}{\ln a} &= \log_a e, \end{aligned}$$

we can replace $\log_a e$ to get

$$\frac{d}{dx}\log_a x = \frac{1}{x\ln a}.$$

You may if you wish memorize the formulas.

Derivative Formulas for a^x and $\log_a x$

$$\frac{d}{dx}a^x = (\ln a)a^x \quad \text{and} \quad \frac{d}{dx}\log_a x = \frac{1}{x\ln a}.$$

Because the "trick" $a = e^{\ln a}$ is often useful, and sometimes essential, it may be better to remember the trick, not the formula.

Example 1.14: Derivative of Exponential Function

Compute the derivative of $f(x) = 2^x$.

Solution.

$$\begin{aligned}\frac{d}{dx}2^x &= \frac{d}{dx}(e^{\ln 2})^x \\ &= \frac{d}{dx}e^{x\ln 2} \\ &= \left(\frac{d}{dx}x\ln 2\right)e^{x\ln 2} \\ &= (\ln 2)e^{x\ln 2} = 2^x\ln 2\end{aligned}$$

♣

Example 1.15: Derivative of Exponential Function

Compute the derivative of $f(x) = 2^{x^2} = 2^{(x^2)}$.

Solution.

$$\begin{aligned}\frac{d}{dx}2^{x^2} &= \frac{d}{dx}e^{x^2\ln 2} \\ &= \left(\frac{d}{dx}x^2\ln 2\right)e^{x^2\ln 2} \\ &= (2\ln 2)xe^{x^2\ln 2} \\ &= (2\ln 2)x2^{x^2}\end{aligned}$$

♣

Example 1.16: Power Rule

Recall that we have not justified the power rule except when the exponent is a positive or negative integer.

Solution. We can use the exponential function to take care of other exponents.

$$\begin{aligned}\frac{d}{dx}x^r &= \frac{d}{dx}e^{r\ln x}\\ &= \left(\frac{d}{dx}r\ln x\right)e^{r\ln x}\\ &= (r\frac{1}{x})x^r\\ &= rx^{r-1}\end{aligned}$$

♣

Exercises for Section 1.5

Find the derivatives of the functions.

Exercise 1.5.1 3^{x^2}

Exercise 1.5.2 $\frac{\sin x}{e^x}$

Exercise 1.5.3 $(e^x)^2$

Exercise 1.5.4 $\sin(e^x)$

Exercise 1.5.5 $e^{\sin x}$

Exercise 1.5.6 $x^{\sin x}$

Exercise 1.5.7 $x^3 e^x$

Exercise 1.5.8 $x+2^x$

Exercise 1.5.9 $(1/3)^{x^2}$

Exercise 1.5.10 e^{4x}/x

Exercise 1.5.11 $\ln(x^3+3x)$

Exercise 1.5.12 $\ln(\cos(x))$

Exercise 1.5.13 $\sqrt{\ln(x^2)}/x$

Exercise 1.5.14 $\ln(\sec(x)+\tan(x))$

Exercise 1.5.15 $x^{\cos(x)}$

Exercise 1.5.16 $x\ln x$

Exercise 1.5.17 $\ln(\ln(3x))$

Exercise 1.5.18 $\frac{1+\ln(3x^2)}{1+\ln(4x)}$

Exercise 1.5.19 *Find the value of a so that the tangent line to $y=\ln(x)$ at $x=a$ is a line through the origin. Sketch the resulting situation.*

Exercise 1.5.20 *If* $f(x) = \ln(x^3 + 2)$ *compute* $f'(e^{1/3})$.

1.6 Logarithmic Differentiation

Previously we've seen how to do the derivative of a number to a function $(a^{f(x)})'$, and also a function to a number $[(f(x))^n]'$. But what about the derivative of a function to a function $[(f(x))^{g(x)}]'$?

In this case, we use a procedure known as **logarithmic differentiation**.

Steps for Logarithmic Differentiation

- Take ln of both sides of $y = f(x)$ to get $\ln y = \ln f(x)$ and simplify using logarithm properties,
- Differentiate implicitly with respect to x and solve for $\frac{dy}{dx}$,
- Replace y with its function of x (i.e., $f(x)$).

Example 1.17: Logarithmic Differentiation

Differentiate $y = x^x$.

Solution. We take ln of both sides:

$$\ln y = \ln x^x.$$

Using log properties we have:

$$\ln y = x \ln x.$$

Differentiating implicitly gives:

$$\frac{y'}{y} = (1)\ln x + x\frac{1}{x}.$$

$$\frac{y'}{y} = \ln x + 1.$$

Solving for y' gives:

$$y' = y(1 + \ln x).$$

Replace $y = x^x$ gives:

$$y' = x^x(1 + \ln x).$$

Another method to find this derivative is as follows:

$$\begin{aligned} \frac{d}{dx}x^x &= \frac{d}{dx}e^{x\ln x} \\ &= \left(\frac{d}{dx}x\ln x\right)e^{x\ln x} \end{aligned}$$

$$
\begin{aligned}
&= (x\frac{1}{x}+\ln x)x^x \\
&= (1+\ln x)x^x
\end{aligned}
$$

♣

In fact, logarithmic differentiation can be used on more complicated products and quotients (not just when dealing with functions to the power of functions).

Example 1.18: Logarithmic Differentiation

Differentiate (assuming $x > 0$):

$$y = \frac{(x+2)^3(2x+1)^9}{x^8(3x+1)^4}.$$

Solution. Using product and quotient rules for this problem is a complete nightmare! Let's apply logarithmic differentiation instead. Take ln of both sides:

$$\ln y = \ln\left(\frac{(x+2)^3(2x+1)^9}{x^8(3x+1)^4}\right).$$

Applying log properties:

$$\ln y = \ln\left((x+2)^3(2x+1)^9\right) - \ln\left(x^8(3x+1)^4\right).$$

$$\ln y = \ln\left((x+2)^3\right) + \ln\left((2x+1)^9\right) - \left[\ln\left(x^8\right) + \ln\left((3x+1)^4\right)\right].$$

$$\ln y = 3\ln(x+2) + 9\ln(2x+1) - 8\ln x - 4\ln(3x+1).$$

Now, differentiating implicitly with respect to x gives:

$$\frac{y'}{y} = \frac{3}{x+2} + \frac{18}{2x+1} - \frac{8}{x} - \frac{12}{3x+1}.$$

Solving for y' gives:

$$y' = y\left(\frac{3}{x+2} + \frac{18}{2x+1} - \frac{8}{x} - \frac{12}{3x+1}\right).$$

Replace $y = \frac{(x+2)^3(2x+1)^9}{x^8(3x+1)^4}$ gives:

$$y' = \frac{(x+2)^3(2x+1)^9}{x^8(3x+1)^4}\left(\frac{3}{x+2} + \frac{18}{2x+1} - \frac{8}{x} - \frac{12}{3x+1}\right).$$

♣

Exercises for Section 1.6

Exercise 1.6.1 *Differentiate the function* $y = \dfrac{(x-1)^8(x-23)^{1/2}}{27x^6(4x-6)^8}$

Exercise 1.6.2 *Differentiate the function* $f(x) = (x+1)^{\sin x}$.

Exercise 1.6.3 *Differentiate the function* $g(x) = \dfrac{e^x(\cos x+2)^3}{\sqrt{x^2+4}}$.

1.7 Hyperbolic Functions

The **hyperbolic functions** are a set of functions that have many applications to mathematics, physics, and engineering. Among many other applications, they are used to describe the formation of satellite rings around planets, to describe the shape of a rope hanging from two points, and have application to the theory of special relativity. This section defines the hyperbolic functions and describes many of their properties, especially their usefulness to calculus.

These functions are sometimes referred to as the "hyperbolic trigonometric functions" as there are many, many connections between them and the standard trigonometric functions. Figure 1.3 demonstrates one such connection. Just as cosine and sine are used to define points on the circle defined by $x^2+y^2=1$, the functions **hyperbolic cosine** and **hyperbolic sine** are used to define points on the hyperbola $x^2-y^2=1$.

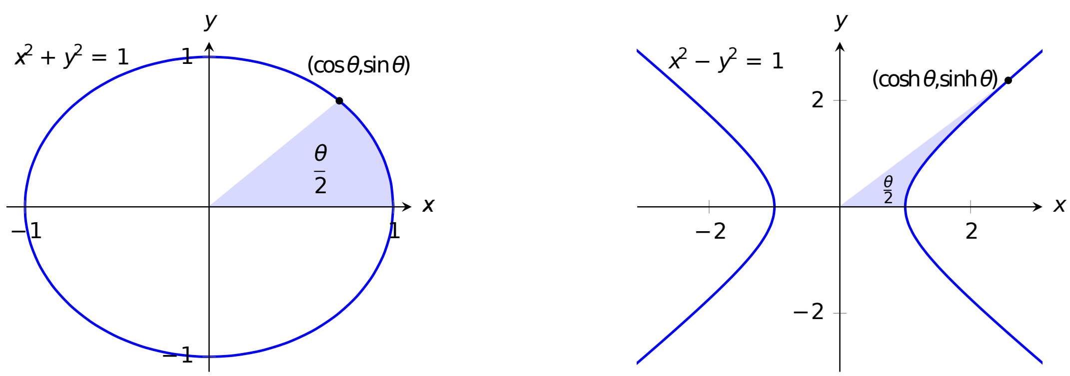


Figure 1.3: Using trigonometric functions to define points on a circle and hyperbolic functions to define points on a hyperbola. The area of the shaded regions are included in them.

We begin with their definition.

This section was adapted by Lyryx from the section "Hyperbolic Functions" in APEX Calculus, Version 3.0, written by G. Hartman. This material is released under Creative Commons license CC BY-NC (`https://creativecommons.org/licenses/by-nc/4.0/`). See the Copyright and Revision History pages in the front of this text for more information.

Definition 1.19: Hyperbolic Functions

1. $\cosh x = \dfrac{e^x + e^{-x}}{2}$
2. $\sinh x = \dfrac{e^x - e^{-x}}{2}$
3. $\tanh x = \dfrac{\sinh x}{\cosh x}$
4. $\operatorname{sech} x = \dfrac{1}{\cosh x}$
5. $\operatorname{csch} x = \dfrac{1}{\sinh x}$
6. $\coth x = \dfrac{\cosh x}{\sinh x}$

These hyperbolic functions are graphed in Figure 1.4. In the graphs of $\cosh x$ and $\sinh x$, graphs of $e^x/2$ and $e^{-x}/2$ are included with dashed lines. As x gets "large," $\cosh x$ and $\sinh x$ each act like $e^x/2$; when x is a large negative number, $\cosh x$ acts like $e^{-x}/2$ whereas $\sinh x$ acts like $-e^{-x}/2$.

Notice the domains of $\tanh x$ and $\operatorname{sech} x$ are $(-\infty, \infty)$, whereas both $\coth x$ and $\operatorname{csch} x$ have vertical asymptotes at $x = 0$. Also note the ranges of these functions, especially $\tanh x$: as $x \to \infty$, both $\sinh x$ and $\cosh x$ approach $e^{-x}/2$, hence $\tanh x$ approaches 1.

The following example explores some of the properties of these functions that bear remarkable resemblance to the properties of their trigonometric counterparts.

Pronunciation Note

"cosh" rhymes with "gosh," "sinh" rhymes with "pinch," and "tanh" rhymes with "ranch."

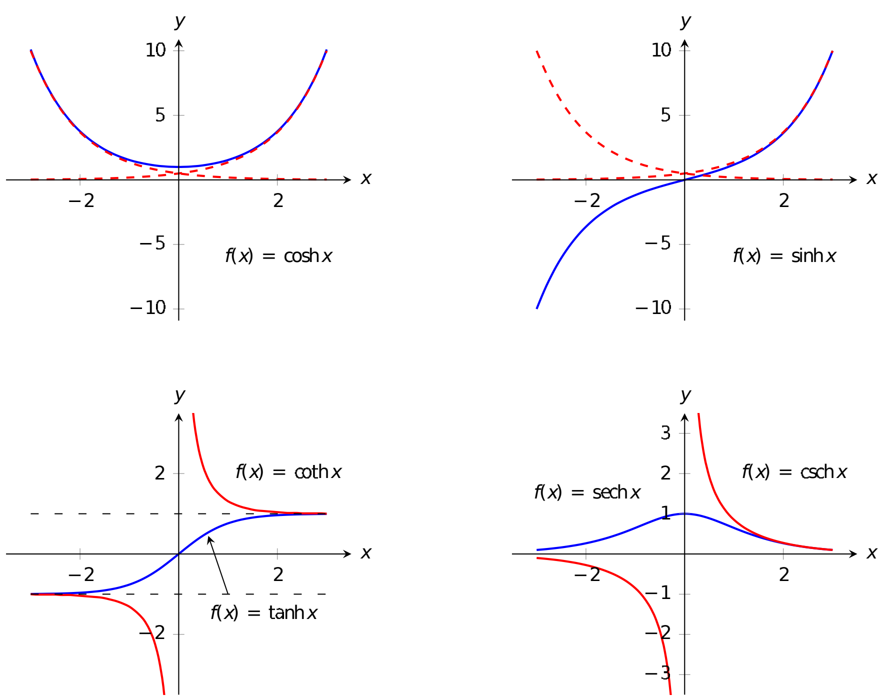


Figure 1.4: Graphs of the hyperbolic functions.

Example 1.20: Exploring Properties of Hyperbolic Functions

Use Definition 1.19 to rewrite the following expressions.

1. $\cosh^2 x - \sinh^2 x$
2. $\tanh^2 x + \operatorname{sech}^2 x$
3. $2\cosh x \sinh x$
4. $\frac{d}{dx}\left(\cosh x\right)$
5. $\frac{d}{dx}\left(\sinh x\right)$
6. $\frac{d}{dx}\left(\tanh x\right)$

Solution.

1.
$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2\\ &= \frac{e^{2x} + 2e^x e^{-x} + e^{-2x}}{4} - \frac{e^{2x} - 2e^x e^{-x} + e^{-2x}}{4}\\ &= \frac{4}{4} = 1.\end{aligned}$$

So $\cosh^2 x - \sinh^2 x = 1$.

2.
$$\begin{aligned}\tanh^2 x + \operatorname{sech}^2 x &= \frac{\sinh^2 x}{\cosh^2 x} + \frac{1}{\cosh^2 x}\\ &= \frac{\sinh^2 x + 1}{\cosh^2 x} && \text{Now use identity from \#1.}\\ &= \frac{\cosh^2 x}{\cosh^2 x} = 1.\end{aligned}$$

So $\tanh^2 x + \operatorname{sech}^2 x = 1$.

3.
$$\begin{aligned}2\cosh x \sinh x &= 2\left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^x - e^{-x}}{2}\right)\\ &= 2 \cdot \frac{e^{2x} - e^{-2x}}{4}\\ &= \frac{e^{2x} - e^{-2x}}{2} = \sinh(2x).\end{aligned}$$

Thus $2\cosh x \sinh x = \sinh(2x)$.

4.
$$\begin{aligned}\frac{d}{dx}\left(\cosh x\right) &= \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right)\\ &= \frac{e^x - e^{-x}}{2}\end{aligned}$$

$$= \sinh x.$$

So $\frac{d}{dx}(\cosh x) = \sinh x.$

5.
$$\begin{aligned}\frac{d}{dx}(\sinh x) &= \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right)\\ &= \frac{e^x + e^{-x}}{2}\\ &= \cosh x.\end{aligned}$$

So $\frac{d}{dx}(\sinh x) = \cosh x.$

6.
$$\begin{aligned}\frac{d}{dx}(\tanh x) &= \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right)\\ &= \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x}\\ &= \frac{1}{\cosh^2 x}\\ &= \operatorname{sech}^2 x.\end{aligned}$$

So $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x.$

♣

The following properties summarize many of the important identities relating to hyperbolic functions. Each can be verified by referring back to Definition 1.19.

Useful Hyperbolic Function Properties

Basic Identities

1. $\cosh^2 x - \sinh^2 x = 1$
2. $\tanh^2 x + \operatorname{sech}^2 x = 1$
3. $\coth^2 x - \operatorname{csch}^2 x = 1$
4. $\cosh 2x = \cosh^2 x + \sinh^2 x$
5. $\sinh 2x = 2\sinh x \cosh x$
6. $\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$
7. $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

Derivatives

1. $\frac{d}{dx}(\cosh x) = \sinh x$
2. $\frac{d}{dx}(\sinh x) = \cosh x$
3. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$
4. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$
5. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$
6. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

Integrals

1. $\int \cosh x \, dx = \sinh x + C$
2. $\int \sinh x \, dx = \cosh x + C$
3. $\int \tanh x \, dx = \ln(\cosh x) + C$
4. $\int \coth x \, dx = \ln|\sinh x| + C$

The following example explores the above properties.

Example 1.21: Derivatives and Integrals of Hyperbolic Functions

Evaluate the following derivatives and integrals.

1. $\dfrac{d}{dx}\big(\cosh 2x\big)$
2. $\displaystyle\int \operatorname{sech}^2(7t-3)\,dt$
3. $\displaystyle\int_0^{\ln 2} \cosh x\,dx$

Solution.

1. Using the Chain Rule directly, we have $\frac{d}{dx}\big(\cosh 2x\big) = 2\sinh 2x$.

   Just to demonstrate that it works, let's also use the Basic Identity from above: $\cosh 2x = \cosh^2 x + \sinh^2 x$.

$$\begin{aligned}\frac{d}{dx}\big(\cosh 2x\big) = \frac{d}{dx}\big(\cosh^2 x + \sinh^2 x\big) &= 2\cosh x \sinh x + 2\sinh x \cosh x \\ &= 4\cosh x \sinh x.\end{aligned}$$

   Using another Basic Identity, we can see that $4\cosh x \sinh x = 2\sinh 2x$. We get the same answer either way.

2. We employ substitution, with $u = 7t-3$ and $du = 7dt$. Applying properties from above we have:

$$\int \operatorname{sech}^2(7t-3)\,dt = \frac{1}{7}\tanh(7t-3) + C.$$

3.
$$\int_0^{\ln 2} \cosh x\,dx = \sinh x\Big|_0^{\ln 2} = \sinh(\ln 2) - \sinh 0 = \sinh(\ln 2).$$

   We can simplify this last expression as $\sinh x$ is based on exponentials:

$$\sinh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} = \frac{2-1/2}{2} = \frac{3}{4}.$$

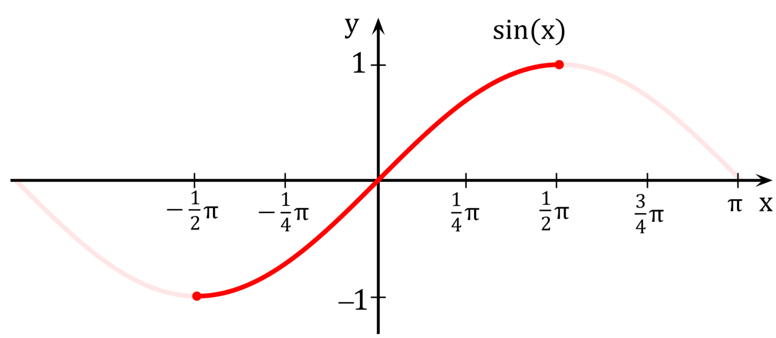
♣

Unit 2: Inverse Trigonometric and Hyperbolic Functions; L'Hopital's Rule

2.1 Inverse Trigonometric Functions

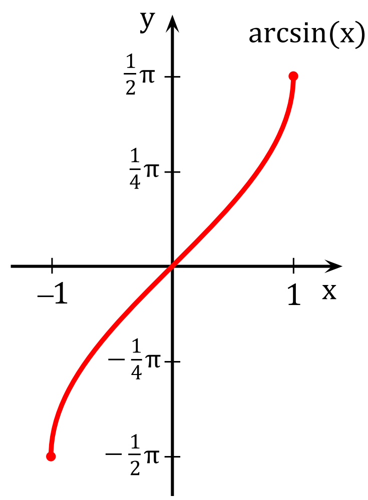
The trigonometric functions frequently arise in problems, and often it is necessary to invert the functions, for example, to find an angle with a specified sine. Of course, there are many angles with the same sine, so the sine function doesn't actually have an inverse that reliably "undoes" the sine function. If you know that $\sin x = 0.5$, you can't reverse this to discover x, that is, you can't solve for x, as there are infinitely many angles with sine 0.5. Nevertheless, it is useful to have something like an inverse to the sine, however imperfect. The usual approach is to pick out some collection of angles that produce all possible values of the sine exactly once. If we "discard" all other angles, the resulting function does have a proper inverse.

The sine takes on all values between -1 and 1 exactly once on the interval $[-\pi/2, \pi/2]$.



If we truncate the sine, keeping only the interval $[-\pi/2, \pi/2]$, then this truncated sine has an inverse function. We call this the inverse sine or the arcsine, and write it in one of two common notation:

$$y = \arcsin(x), \text{ or } y = \sin^{-1}(x)$$



Recall that a function and its inverse undo each other in either order, for example, $(\sqrt[3]{x})^3 = x$ and $\sqrt[3]{x^3} = x$. This does not work with the sine and the "inverse sine" because the inverse sine is the inverse of the truncated sine function, not the real sine function. It is true that $\sin(\arcsin(x)) = x$, that is, the sine undoes the arcsine. It is not true that the arcsine undoes the sine, for example, $\sin(5\pi/6) = 1/2$ and $\arcsin(1/2) = \pi/6$, so doing first the sine then the arcsine does not get us back where we started. This is because $5\pi/6$ is not in the domain of the truncated sine. If we start with an angle between $-\pi/2$ and $\pi/2$ then the arcsine does reverse the sine: $\sin(\pi/6) = 1/2$ and $\arcsin(1/2) = \pi/6$.

Example 2.22: Arcsine of Common Values

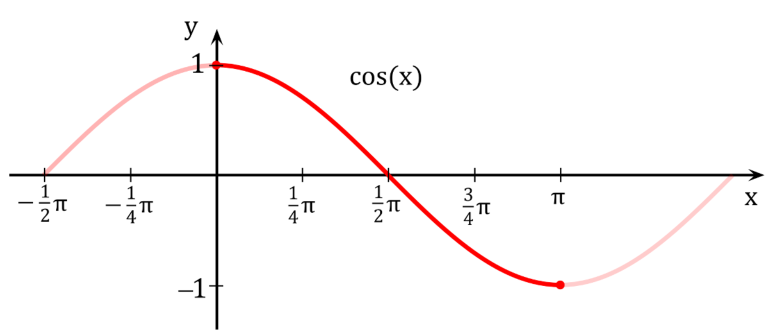
Compute $\sin^{-1}(0)$, $\sin^{-1}(1)$ *and* $\sin^{-1}(-1)$.

Solution. These come directly from the graph of $y = \arcsin x$:

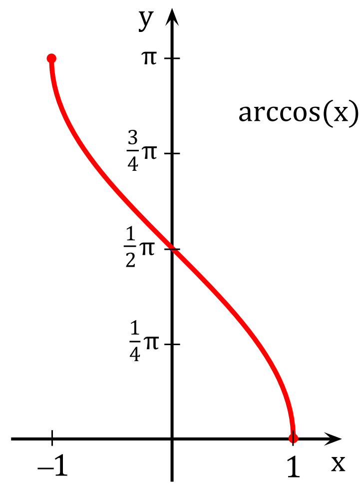
$$\sin^{-1}(0) = 0 \qquad \sin^{-1}(1) = \frac{\pi}{2} \qquad \sin^{-1}(-1) = -\frac{\pi}{2}$$

♣

We can do something similar for the cosine function. As with the sine, we must first truncate the cosine so that it can be inverted, in particular, we use the interval $[0,\pi]$.



Note that the truncated cosine uses a different interval than the truncated sine, so that if $y = \arccos(x)$ we know that $0 \le y \le \pi$.



Example 2.23: Arccosine of Common Values

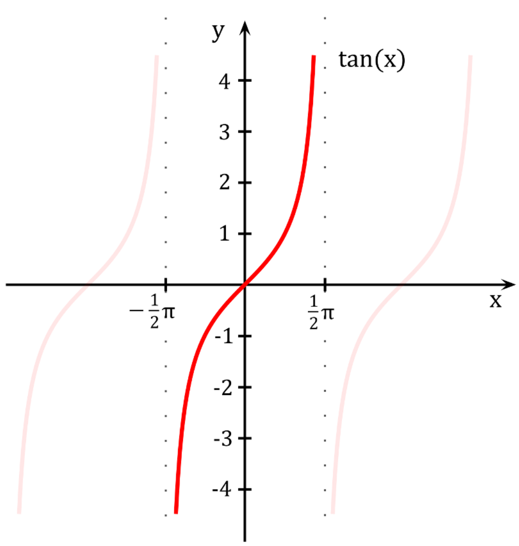
Compute $\cos^{-1}(0)$, $\cos^{-1}(1)$ *and* $\cos^{-1}(-1)$.

Solution. These come directly from the graph of $y = \arccos x$:

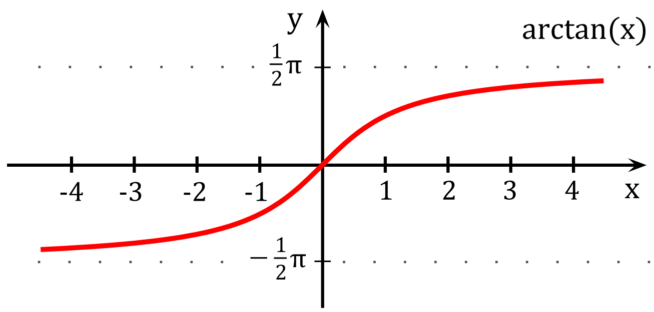
$$\cos^{-1}(0) = \frac{\pi}{2} \qquad \cos^{-1}(1) = 0 \qquad \cos^{-1}(-1) = \pi$$

♣

Finally we look at the tangent; the other trigonometric functions also have "partial inverses" but the sine, cosine and tangent are enough for most purposes. The truncated tangent uses an interval of $(-\pi/2, \pi/2)$.



Reflecting the truncated tangent in the line $y = x$ gives the arctangent function.



Example 2.24: Arctangent of Common Values

Compute $\tan^{-1}(0)$. *What value does* $\tan^{-1} x$ *approach as* x *gets larger and larger? What value does* $\tan^{-1} x$ *approach as* x *gets large (and negative)?*

Solution. These come directly from the graph of $y = \arctan x$. In particular, $\tan^{-1}(0) = 0$. As x gets larger and larger, $\tan^{-1} x$ approaches a value of $\frac{\pi}{2}$, whereas, as x gets large but negative, $\tan^{-1} x$ approaches a value of $-\frac{\pi}{2}$. ♣

The cancellation rules are tricky since we restricted the domains of the trigonometric functions in order to obtain inverse trig functions:

Cancellation Rules

$$\sin(\sin^{-1} x) = x, \quad x \in [-1, 1] \qquad \sin^{-1}(\sin x) = x, \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\cos(\cos^{-1} x) = x, \quad x \in [-1, 1] \qquad \cos^{-1}(\cos x) = x, \quad x \in [0, \pi]$$

$$\tan(\tan^{-1} x) = x, \quad x \in (-\infty, \infty) \qquad \tan^{-1}(\tan x) = x, \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Example 2.25: Arcsine of $1/2$

Find $\sin^{-1}(1/2)$.

Solution. Since $\sin^{-1}(x)$ outputs values in $[-\pi/2, \ \pi/2]$, the answer must be in this interval. Let $\theta = \sin^{-1}(1/2)$. We need to compute θ. Take the sine of both sides to get $\sin\theta = \sin(\sin^{-1}(1/2)) = 1/2$ by the cancellation rule. There are many angles θ that work, but we want the one in the interval $[-\pi/2, \ \pi/2]$. Thus, $\theta = \pi/6$ and hence, $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$. ♣

Example 2.26: Arccosine and the Cancellation Rule

Compute $\cos^{-1}(\cos(0))$, $\cos^{-1}(\cos(\pi))$, $\cos^{-1}(\cos(2\pi))$, $\cos^{-1}(\cos(3\pi))$.

Solution. Since $\cos^{-1}(x)$ outputs values in $[0,\ \pi]$, the answers must be in this interval. The first two we can cancel using the cancellation rules:

$$\cos^{-1}(\cos(0)) = 0 \qquad \text{and} \qquad \cos^{-1}(\cos(\pi)) = \pi.$$

The third one we cannot cancel since $2\pi \notin [0,\pi]$:

$$\cos^{-1}(\cos(2\pi)) \quad \text{is NOT equal to} \quad 2\pi.$$

But we know that cosine is a 2π-periodic function, so $\cos(2\pi) = \cos(0)$:

$$\cos^{-1}(\cos(2\pi)) = \cos^{-1}(\cos(0)) = 0$$

Similarly with the fourth one, we can NOT cancel yet since $3\pi \notin [0,\pi]$. Using $\cos(3\pi) = \cos(3\pi - 2\pi) = \cos(\pi)$:

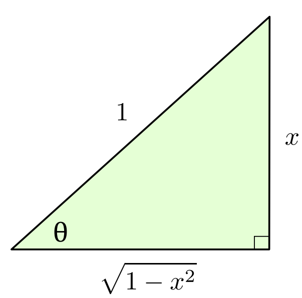
$$\cos^{-1}(\cos(3\pi)) = \cos^{-1}(\cos(\pi)) = \pi.$$

♣

Example 2.27: The Triangle Technique

Rewrite the expression $\cos(\sin^{-1} x)$ *without trig functions. Note that the domain of this function is all* $x \in [-1,1]$.

Solution. Let $\theta = \sin^{-1} x$. We need to compute $\cos\theta$. Taking the sine of both sides gives $\sin\theta = \sin(\sin^{-1}(x)) = x$ by the cancellation rule. We then draw a right triangle using $\sin\theta = x/1$:



If z is the remaining side, then by the Pythagorean Theorem:

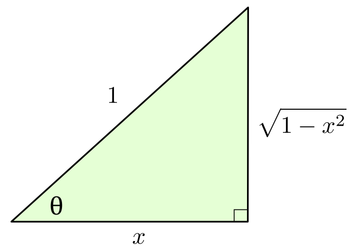
$$z^2 + x^2 = 1 \qquad \to \qquad z^2 = 1 - x^2 \qquad \to \qquad z = \pm\sqrt{1-x^2}$$

and hence $z = +\sqrt{1-x^2}$ since $\theta \in [-\pi/2,\ \pi/2]$. Thus, $\cos\theta = \sqrt{1-x^2}$ by SOH CAH TOA, so, $\cos(\sin^{-1} x) = \sqrt{1-x^2}$. ♣

Example 2.28: The Triangle Technique 2

For $x \in (0,1)$, *rewrite the expression* $\sin(2\cos^{-1} x)$. *Compute* $\sin(2\cos^{-1}(1/2))$.

Solution. Let $\theta = \cos^{-1} x$ so that $\cos\theta = x$. The question now asks for us to compute $\sin(2\theta)$. We then draw a right triangle using $\cos\theta = x/1$:



To find $\sin(2\theta)$ we use the double angle formula $\sin(2\theta) = 2\sin\theta\cos\theta$. But $\sin\theta = \sqrt{1-x^2}$, for $\theta \in [0,\pi]$, and $\cos\theta = x$. Therefore, $\sin(2\cos^{-1} x) = 2x\sqrt{1-x^2}$. When $x = 1/2$ we have $\sin(2\cos^{-1}(1/2)) = \dfrac{\sqrt{3}}{2}$. ♣

Exercises for 2.1

Exercise 2.1.1 *Compute the following:*

(a) $\sin^{-1}(\sqrt{3}/2)$

(b) $\cos^{-1}(-\sqrt{2}/2)$

Exercise 2.1.2 *Compute the following:*

(a) $\sin^{-1}(\sin(\pi/4))$

(b) $\sin^{-1}(\sin(17\pi/3))$

(c) $\cos\left(\cos^{-1}(1/3)\right)$

(d) $\tan\left(\cos^{-1}(-4/5)\right)$

Exercise 2.1.3 *Rewrite the expression* $\tan\left(\cos^{-1} x\right)$ *without trigonometric functions. What is the domain of this function?*

2.2 Inverse Hyperbolic Functions

Just as the inverse trigonometric functions are useful in certain integrations, the inverse hyperbolic functions are useful with others. Table 2.1 below shows the restrictions on the domains to make each function one-to-one and the resulting domains and ranges of their inverse functions. Their graphs are shown in Figure 2.5.

Because the hyperbolic functions are defined in terms of exponential functions, their inverses can be expressed in terms of logarithms as shown in the earlier properties. It is often more convenient to refer to $\sinh^{-1} x$ than to $\ln\left(x+\sqrt{x^2+1}\right)$, especially when one is working on theory and does not need to compute actual values. On the other hand, when computations are needed, technology is often helpful but many hand-held calculators lack a *convenient* $\sinh^{-1} x$ button. (Often it can be accessed under a menu system, but not conveniently.) In such a situation, the logarithmic representation is useful. The reader is not encouraged to memorize these, but rather know they exist and know how to use them when needed.

Function	Domain	Range	Function	Domain	Range
$\cosh x$	$[0,\infty)$	$[1,\infty)$	$\cosh^{-1} x$	$[1,\infty)$	$[0,\infty)$
$\sinh x$	$(-\infty,\infty)$	$(-\infty,\infty)$	$\sinh^{-1} x$	$(-\infty,\infty)$	$(-\infty,\infty)$
$\tanh x$	$(-\infty,\infty)$	$(-1,1)$	$\tanh^{-1} x$	$(-1,1)$	$(-\infty,\infty)$
$\operatorname{sech} x$	$[0,\infty)$	$(0,1]$	$\operatorname{sech}^{-1} x$	$(0,1]$	$[0,\infty)$
$\operatorname{csch} x$	$(-\infty,0)\cup(0,\infty)$	$(-\infty,0)\cup(0,\infty)$	$\operatorname{csch}^{-1} x$	$(-\infty,0)\cup(0,\infty)$	$(-\infty,0)\cup(0,\infty)$
$\coth x$	$(-\infty,0)\cup(0,\infty)$	$(-\infty,-1)\cup(1,\infty)$	$\coth^{-1} x$	$(-\infty,-1)\cup(1,\infty)$	$(-\infty,0)\cup(0,\infty)$

Table 2.1: Domains and ranges of the hyperbolic and inverse hyperbolic functions.

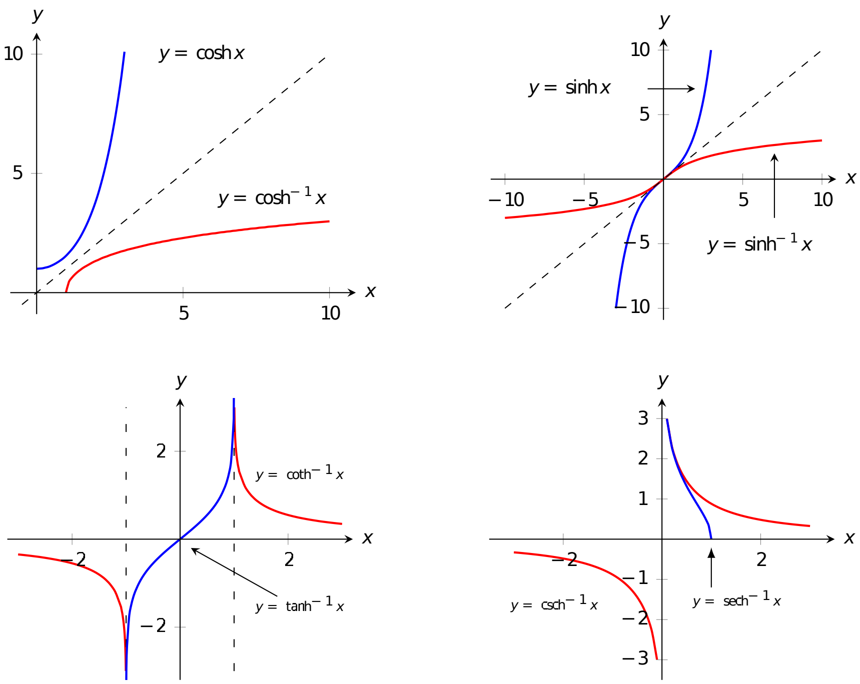


Figure 2.5: Graphs of the hyperbolic functions and their inverses.

Logarithmic Definitions of Inverse Hyperbolic Functions

1. $\cosh^{-1}x = \ln\left(x+\sqrt{x^2-1}\right);\ x \geq 1$
2. $\tanh^{-1}x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right);\ |x| < 1$
3. $\operatorname{sech}^{-1}x = \ln\left(\dfrac{1+\sqrt{1-x^2}}{x}\right);\ 0 < x \leq 1$
4. $\sinh^{-1}x = \ln\left(x+\sqrt{x^2+1}\right)$
5. $\coth^{-1}x = \dfrac{1}{2}\ln\left(\dfrac{x+1}{x-1}\right);\ |x| > 1$
6. $\operatorname{csch}^{-1}x = \ln\left(\dfrac{1}{x}+\dfrac{\sqrt{1+x^2}}{|x|}\right);\ x \neq 0$

The following boxes highlight the derivatives and integrals of inverse hyperbolic functions.

Derivatives Involving Inverse Hyperbolic Functions

1. $\dfrac{d}{dx}\left(\cosh^{-1}x\right) = \dfrac{1}{\sqrt{x^2-1}};\ x > 1$
2. $\dfrac{d}{dx}\left(\sinh^{-1}x\right) = \dfrac{1}{\sqrt{x^2+1}}$
3. $\dfrac{d}{dx}\left(\tanh^{-1}x\right) = \dfrac{1}{1-x^2};\ |x| < 1$
4. $\dfrac{d}{dx}\left(\operatorname{sech}^{-1}x\right) = \dfrac{-1}{x\sqrt{1-x^2}};0 < x < 1$
5. $\dfrac{d}{dx}\left(\operatorname{csch}^{-1}x\right) = \dfrac{-1}{|x|\sqrt{1+x^2}};\ x \neq 0$
6. $\dfrac{d}{dx}\left(\coth^{-1}x\right) = \dfrac{1}{1-x^2};\ |x| > 1$

Integrals Involving Inverse Hyperbolic Functions

1. $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}\left(\frac{x}{a}\right)+C;\ 0<a<x \quad = \ln\left|x+\sqrt{x^2-a^2}\right|+C$

2. $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right)+C;\ a>0 \quad = \ln\left|x+\sqrt{x^2+a^2}\right|+C$

3. $\displaystyle\int \frac{1}{a^2-x^2}\,dx = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right)+C & x^2<a^2 \\ \frac{1}{a}\coth^{-1}\left(\frac{x}{a}\right)+C & a^2<x^2 \end{cases} \quad = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|+C$

4. $\displaystyle\int \frac{1}{x\sqrt{a^2-x^2}}\,dx = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{x}{a}\right)+C;\ 0<x<a \quad = \frac{1}{a}\ln\left(\frac{x}{a+\sqrt{a^2-x^2}}\right)+C$

5. $\displaystyle\int \frac{1}{x\sqrt{x^2+a^2}}\,dx = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{x}{a}\right|+C;\ x\neq 0,\ a>0 = \frac{1}{a}\ln\left|\frac{x}{a+\sqrt{a^2+x^2}}\right|+C$

We practice using the derivative and integral formulas in the following example.

Example 2.29: Derivatives and Integrals Involving Inverse Hyperbolic Functions

Evaluate the following.

1. $\displaystyle\frac{d}{dx}\left[\cosh^{-1}\left(\frac{3x-2}{5}\right)\right]$

2. $\displaystyle\int \frac{1}{x^2-1}\,dx$

3. $\displaystyle\int \frac{1}{\sqrt{9x^2+10}}\,dx$

Solution.

1. Using the above derivative definitions together with the Chain Rule gives:

$$\frac{d}{dx}\left[\cosh^{-1}\left(\frac{3x-2}{5}\right)\right] = \frac{1}{\sqrt{\left(\frac{3x-2}{5}\right)^2-1}}\times\frac{3}{5}.$$

2. Multiplying the numerator and denominator by (-1) gives: $\displaystyle\int \frac{1}{x^2-1}\,dx = \int \frac{-1}{1-x^2}\,dx$. The second integral can be solved with a direct application of the above definitions, with $a=1$. Thus

$$\begin{aligned}\int \frac{1}{x^2-1}\,dx &= -\int \frac{1}{1-x^2}\,dx \\ &= \begin{cases} -\tanh^{-1}(x)+C & x^2<1 \\ -\coth^{-1}(x)+C & 1<x^2 \end{cases} \\ &= -\frac{1}{2}\ln\left|\frac{x+1}{x-1}\right|+C \\ &= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+C. \end{aligned} \tag{2.1}$$

3. This requires a substitution, then application of an integral definition from above.

   Let $u = 3x$, hence $du = 3dx$. We have

$$\int \frac{1}{\sqrt{9x^2+10}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{u^2+10}}\,du.$$

   Note $a^2 = 10$, hence $a = \sqrt{10}$. Now apply the integral rule.

$$\begin{aligned} &= \frac{1}{3}\sinh^{-1}\left(\frac{3x}{\sqrt{10}}\right) + C \\ &= \frac{1}{3}\ln\left|3x + \sqrt{9x^2+10}\right| + C. \end{aligned}$$

♣

This section covers a lot of ground. New functions were introduced, along with some of their fundamental identities, their derivatives and antiderivatives, their inverses, and the derivatives and antiderivatives of these inverses.

Do not view this section as containing a source of information to be memorized, but rather as a reference for future problem solving. The information highlighted in boxes above is especially important.

Exercises for Section 2.2

Exercise 2.2.1 *In this section, the equation* $\int \tanh x\,dx = \ln(\cosh x) + C$ *is given. Why is "*$\ln|\cosh x|$*" not used – i.e., why are absolute values not necessary?*

Exercise 2.2.2 *The hyperbolic functions are used to define points on the right hand portion of the hyperbola* $x^2 - y^2 = 1$*, as shown in Figure 1.3. How can we use the hyperbolic functions to define points on the left hand portion of the hyperbola?*

In the following exercises, verify the given identity using Definition 1.19, as done in Example 1.20.

Exercise 2.2.3 $\coth^2 x - \operatorname{csch}^2 x = 1$

Exercise 2.2.4 $\cosh 2x = \cosh^2 x + \sinh^2 x$

Exercise 2.2.5 $\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$

Exercise 2.2.6 $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

Exercise 2.2.7 $\dfrac{d}{dx}[\operatorname{sech} x] = -\operatorname{sech} x \tanh x$

Exercise 2.2.8 $\dfrac{d}{dx}[\coth x] = -\operatorname{csch}^2 x$

Exercise 2.2.9 $\int \tanh x\,dx = \ln(\cosh x) + C$

Exercise 2.2.10 $\int \coth x\,dx = \ln|\sinh x| + C$

In the following exercises, find the derivative of the given function.

Exercise 2.2.11 $f(x) = \cosh 2x$

Exercise 2.2.12 $f(x) = \tanh(x^2)$

Exercise 2.2.13 $f(x) = \ln(\sinh x)$

Exercise 2.2.14 $f(x) = \sinh x \cosh x$

Exercise 2.2.15 $f(x) = x \sinh x - \cosh x$

Exercise 2.2.16 $f(x) = \text{sech}^{-1}(x^2)$

Exercise 2.2.17 $f(x) = \sinh^{-1}(3x)$

Exercise 2.2.18 $f(x) = \cosh^{-1}(2x^2)$

Exercise 2.2.19 $f(x) = \tanh^{-1}(x+5)$

Exercise 2.2.20 $f(x) = \tanh^{-1}(\cos x)$

Exercise 2.2.21 $f(x) = \cosh^{-1}(\sec x)$

In the following exercises, find the equation of the line tangent to the function at the given x-value.

Exercise 2.2.22 $f(x) = \sinh x$ *at* $x = 0$

Exercise 2.2.23 $f(x) = \cosh x$ *at* $x = \ln 2$

Exercise 2.2.24 $f(x) = \text{sech}^2 x$ *at* $x = \ln 3$

Exercise 2.2.25 $f(x) = \sinh^{-1} x$ *at* $x = 0$

Exercise 2.2.26 $f(x) = \cosh^{-1} x$ *at* $x = \sqrt{2}$

In the following exercises, evaluate the given indefinite integral.

Exercise 2.2.27 $\displaystyle\int \tanh(2x)\, dx$

Exercise 2.2.28 $\displaystyle\int \cosh(3x-7)\, dx$

Exercise 2.2.29 $\displaystyle\int \sinh x \cosh x\, dx$

Exercise 2.2.30 $\displaystyle\int x \cosh x\, dx$

Exercise 2.2.31 $\displaystyle\int x \sinh x\, dx$

Exercise 2.2.32 $\displaystyle\int \frac{1}{9-x^2}\, dx$

Exercise 2.2.33 $\displaystyle\int \frac{2x}{\sqrt{x^4-4}}\, dx$

Exercise 2.2.34 $\displaystyle\int \frac{\sqrt{x}}{\sqrt{1+x^3}}\, dx$

Exercise 2.2.35 $\displaystyle\int \frac{1}{x^4-16}\, dx$

Exercise 2.2.36 $\displaystyle\int \frac{1}{x^2+x}\, dx$

Exercise 2.2.37 $\displaystyle\int \frac{e^x}{e^{2x}+1}\, dx$

Exercise 2.2.38 $\displaystyle\int \sinh^{-1} x\, dx$

Exercise 2.2.39 $\displaystyle\int \tanh^{-1} x\, dx$

Exercise 2.2.40 $\displaystyle\int \text{sech} x\, dx$ *(Hint: mutiply by* $\frac{\cosh x}{\cosh x}$*; set* $u = \sinh x$*.)*

In the following exercises, evaluate the given definite integral.

Exercise 2.2.41 $\displaystyle\int_{-1}^{1} \sinh x\, dx$

Exercise 2.2.42 $\displaystyle\int_{-\ln 2}^{\ln 2} \cosh x\, dx$

Exercise 2.2.43 $\displaystyle\int_{0}^{1} \tanh^{-1} x\, dx$

2.3 Additional Exercises

Exercise 2.3.1 *If* $f(x) = \frac{1}{x-1}$, *then which of the following is equal to* $f\left(\frac{1}{x}\right)$?

(a) $f(x)$

(b) $-f(x)$

(c) $xf(x)$

(d) $-xf(x)$

(e) $\frac{f(x)}{x}$

(f) $-\frac{f(x)}{x}$

Exercise 2.3.2 *If* $f(x) = \frac{x}{x+3}$, *then find and simplify* $\frac{f(x)-f(2)}{x-2}$.

Exercise 2.3.3 *If* $f(x) = x^2$, *then find and simplify* $\frac{f(3+h)-f(3)}{h}$.

Exercise 2.3.4 *What is the domain of*

(a) $f(x) = \frac{\sqrt{x-2}}{x^2-9}$?

(b) $g(x) = \frac{\sqrt[3]{x-2}}{x^2-9}$?

Exercise 2.3.5 *Suppose that* $f(x) = x^3$ *and* $g(x) = x$. *What is the domain of* $\frac{f}{g}$?

Exercise 2.3.6 *Suppose that* $f(x) = 3x-4$. *Find a function* g *such that* $(g \circ f)(x) = 5x+2$.

Exercise 2.3.7 *Which of the following functions is one-to-one?*

(a) $f(x) = x^2+4x+3$

(b) $g(x) = |x|+2$

(c) $h(x) = \sqrt[3]{x+1}$

(d) $F(x) = \cos x,\ -\pi \leq x \leq \pi$

(e) $G(x) = e^x + e^{-x}$

Exercise 2.3.8 *What is the inverse of* $f(x) = \ln\left(\frac{e^x}{e^x - 1}\right)$*? What is the domain of* f^{-1}*?*

Exercise 2.3.9 *Solve the following equations.*

(a) $e^{2-x} = 3$

(b) $e^{x^2} = e^{4x-3}$

(c) $\ln(1+\sqrt{x}) = 2$

(d) $\ln(x^2 - 3) = \ln 2 + \ln x$

Exercise 2.3.10 *Find the exact value of* $\sin^{-1}\left(-\sqrt{2}/2\right) - \cos^{-1}\left(-\sqrt{2}/2\right)$.

Exercise 2.3.11 *Find* $\sin^{-1}(\sin(23\pi/5))$.

Exercise 2.3.12 *It can be proved that* $f(x) = x^3 + x + e^{x-1}$ *is one-to-one. What is the value of* $f^{-1}(3)$*?*

Exercise 2.3.13 *Sketch the graph of* $f(x) = \begin{cases} -x & \text{if } x \le 0 \\ \tan^{-1} x & \text{if } x > 0 \end{cases}$

2.4 L'Hôpital's Rule

The following application of derivatives allows us to compute certain limits.

Definition 2.30: Limits of the Indeterminate Forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$

A limit of a quotient $\lim_{x\to a} \frac{f(x)}{g(x)}$ *is said to be an* ***indeterminate form of the type*** $\frac{0}{0}$ *if*

$$\text{both } f(x) \to 0 \text{ and } g(x) \to 0 \text{ as } x \to a.$$

Likewise, it is said to be an ***indeterminate form of the type*** $\frac{\infty}{\infty}$ *if*

$$\text{both } f(x) \to \pm\infty \text{ and } g(x) \to \pm\infty \text{ as } x \to a.$$

Note that the two $\pm$ *signs are independent of each other.*

Theorem 2.31: L'Hôpital's Rule

For a limit $\lim_{x\to a} \frac{f(x)}{g(x)}$ *of the indeterminate form* $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ *exists or equals* ∞ *or* $-\infty$.

This theorem is somewhat difficult to prove, in part because it incorporates so many different possibilities, so we will not prove it here.

We should also note that there may be instances where we would need to apply L'Hôpital's Rule multiple times, but we must confirm that $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ is still indeterminate before we attempt to apply L'Hôpital's Rule again. Finally, we want to mention that L'Hôpital's rule is also valid for one-sided limits and limits at infinity.

Example 2.32: L'Hôpital's Rule

Compute $\displaystyle\lim_{x \to \pi} \frac{x^2 - \pi^2}{\sin x}$.

Solution. We use L'Hôpital's Rule: Since the numerator and denominator both approach zero,

$$\lim_{x \to \pi} \frac{x^2 - \pi^2}{\sin x} = \lim_{x \to \pi} \frac{2x}{\cos x},$$

provided the latter exists. But in fact this is an easy limit, since the denominator now approaches -1, so

$$\lim_{x \to \pi} \frac{x^2 - \pi^2}{\sin x} = \frac{2\pi}{-1} = -2\pi.$$

♣

Example 2.33: L'Hôpital's Rule

Compute $\displaystyle\lim_{x \to \infty} \frac{2x^2 - 3x + 7}{x^2 + 47x + 1}$.

Solution. As x goes to infinity, both the numerator and denominator go to infinity, so we may apply L'Hôpital's Rule:

$$\lim_{x \to \infty} \frac{2x^2 - 3x + 7}{x^2 + 47x + 1} = \lim_{x \to \infty} \frac{4x - 3}{2x + 47}.$$

In the second quotient, it is still the case that the numerator and denominator both go to infinity, so we are allowed to use L'Hôpital's Rule again:

$$\lim_{x \to \infty} \frac{4x - 3}{2x + 47} = \lim_{x \to \infty} \frac{4}{2} = 2.$$

So the original limit is 2 as well. ♣

Example 2.34: L'Hôpital's Rule

Compute $\displaystyle\lim_{x \to 0} \frac{\sec x - 1}{\sin x}$.

Solution. Both the numerator and denominator approach zero, so applying L'Hôpital's Rule:

$$\lim_{x \to 0} \frac{\sec x - 1}{\sin x} = \lim_{x \to 0} \frac{\sec x \tan x}{\cos x} = \frac{1 \cdot 0}{1} = 0.$$

♣

L'Hôpital's rule concerns limits of a quotient that are indeterminate forms. But not all functions are given in the form of a quotient. But all the same, nothing prevents us from re-writing a given function in the form of a quotient. Indeed, some functions whose given form involve either a product $f(x)g(x)$ or a power $f(x)^{g(x)}$ carry indeterminacies such as $0 \cdot \pm\infty$ and $1^{\pm\infty}$. Something small times something numerically large (positive or negative) could be anything. It depends on how small and how large each piece turns out to be. A number close to 1 raised to a numerically large (positive or negative) power could be anything. It depends on how close to 1 the base is, whether the base is larger than or smaller than 1, and how large the exponent is (and its sign). We can use suitable algebraic manipulations to relate them to indeterminate quotients. We will illustrate with two examples, first a product and then a power.

Example 2.35: L'Hôpital's Rule

Compute $\lim_{x\to 0^+} x\ln x$.

Solution. This doesn't appear to be suitable for L'Hôpital's Rule, but it also is not "obvious". As x approaches zero, $\ln x$ goes to $-\infty$, so the product looks like:

$$(\text{something very small}) \cdot (\text{something very large and negative}).$$

This could be anything: it depends on *how small* and *how large* each piece of the function turns out to be. As defined earlier, this is a type of $\pm$"$0 \cdot \infty$", which is indeterminate. So we can in fact apply L'Hôpital's Rule after re-writing it in the form $\frac{\infty}{\infty}$:

$$x\ln x = \frac{\ln x}{1/x} = \frac{\ln x}{x^{-1}}.$$

Now as x approaches zero, both the numerator and denominator approach infinity (one $-\infty$ and one $+\infty$, but only the size is important). Using L'Hôpital's Rule:

$$\lim_{x\to 0^+} \frac{\ln x}{x^{-1}} = \lim_{x\to 0^+} \frac{1/x}{-x^{-2}} = \lim_{x\to 0^+} \frac{1}{x}(-x^2) = \lim_{x\to 0^+} -x = 0.$$

One way to interpret this is that since $\lim_{x\to 0^+} x\ln x = 0$, the x approaches zero much faster than the $\ln x$ approaches $-\infty$.

♣

Finally, we illustrate how a limit of the type "1^∞" can be indeterminate.

Example 2.36: L'Hôpital's Rule

Evaluate $\lim_{x\to 1^+} x^{1/(x-1)}$.

Solution. Plugging in $x = 1$ (from the right) gives a limit of the type "1^∞". To deal with this type of limit we will use logarithms. Let

$$L = \lim_{x\to 1^+} x^{1/(x-1)}.$$

Now, take the natural log of both sides:

$$\ln L = \lim_{x\to 1^+} \ln\left(x^{1/(x-1)}\right).$$

Using log properties we have:

$$\ln L = \lim_{x\to 1^+} \frac{\ln x}{x-1}.$$

The right side limit is now of the type $0/0$, therefore, we can apply L'Hôpital's Rule:

$$\ln L = \lim_{x\to 1^+} \frac{\ln x}{x-1} = \lim_{x\to 1^+} \frac{1/x}{1} = 1$$

Thus, $\ln L = 1$ and hence, our original limit (denoted by L) is: $L = e^1 = e$. That is,

$$L = \lim_{x\to 1^+} x^{1/(x-1)} = e.$$

In this case, even though our limit had a type of "1^∞", it actually had a value of e. ♣

Exercises for 2.4

Compute the following limits.

Exercise 2.4.1 $\displaystyle\lim_{x\to 0} \frac{\cos x - 1}{\sin x}$

Exercise 2.4.2 $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^3}$

Exercise 2.4.5 $\displaystyle\lim_{x\to 0} \frac{\sqrt{9+x}-3}{x}$

Exercise 2.4.6 $\displaystyle\lim_{x\to 2} \frac{2-\sqrt{x+2}}{4-x^2}$

Exercise 2.4.7 $\displaystyle\lim_{x\to 1} \frac{\sqrt{x}-1}{\sqrt[1/3]{x}-1}$

Exercise 2.4.8 $\displaystyle\lim_{x\to 0} \frac{(1-x)^{1/4}-1}{x}$

Exercise 2.4.9 $\displaystyle\lim_{t\to 0} \left(t+\frac{1}{t}\right)\left((4-t)^{3/2}-8\right)$

Exercise 2.4.10 $\displaystyle\lim_{t\to 0^+} \left(\frac{1}{t}+\frac{1}{\sqrt{t}}\right)\left(\sqrt{t+1}-1\right)$

Exercise 2.4.11 $\displaystyle\lim_{x\to 0} \frac{x^2}{\sqrt{2x+1}-1}$

Exercise 2.4.12 $\displaystyle\lim_{u\to 1} \frac{(u-1)^3}{(1/u)-u^2+3/u-3}$

Exercise 2.4.13 $\displaystyle\lim_{x\to 0} \frac{2+(1/x)}{3-(2/x)}$

Exercise 2.4.3 $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x}$

Exercise 2.4.4 $\displaystyle\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$

Exercise 2.4.14 $\displaystyle\lim_{x\to 0^+} \frac{1+5/\sqrt{x}}{2+1/\sqrt{x}}$

Exercise 2.4.15 $\displaystyle\lim_{x\to \pi/2} \frac{\cos x}{(\pi/2)-x}$

Exercise 2.4.16 $\displaystyle\lim_{x\to 0} \frac{e^x-1}{x}$

Exercise 2.4.17 $\displaystyle\lim_{x\to 0} \frac{x^2}{e^x-x-1}$

Exercise 2.4.18 $\displaystyle\lim_{x\to 1} \frac{\ln x}{x-1}$

Exercise 2.4.19 $\displaystyle\lim_{x\to 0} \frac{\ln(x^2+1)}{x}$

Exercise 2.4.20 $\displaystyle\lim_{x\to 1} \frac{x\ln x}{x^2-1}$

Exercise 2.4.21 $\displaystyle\lim_{x\to 0} \frac{\sin(2x)}{\ln(x+1)}$

Exercise 2.4.22 $\displaystyle\lim_{x\to 1} \frac{x^{1/4}-1}{x}$

Exercise 2.4.23 $\displaystyle\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}$

Exercise 2.4.24 $\lim_{x\to 0} \frac{3x^2+x+2}{x-4}$

Exercise 2.4.25 $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{\sqrt{x+4}-2}$

Exercise 2.4.26 $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{\sqrt{x+2}-2}$

Exercise 2.4.27 $\lim_{x\to 0^+} \frac{\sqrt{x+1}+1}{\sqrt{x+1}-1}$

Exercise 2.4.28 $\lim_{x\to 0} \frac{\sqrt{x^2+1}-1}{\sqrt{x+1}-1}$

Exercise 2.4.29 $\lim_{x\to 1} (x+5)\left(\frac{1}{2x}+\frac{1}{x+2}\right)$

Exercise 2.4.30 $\lim_{x\to 2} \frac{x^3-6x-2}{x^3+4}$

Exercise 2.4.31 *Discuss what happens if we try to use L'Hôpital's rule to find the limit* $\lim_{x\to\infty} \frac{x+\sin x}{x+1}$.

Unit 3: Techniques of Integration

3.1 Integration by Parts

We have already seen that recognizing the product rule can be useful, when we noticed that

$$\int \sec^3 u + \sec u \tan^2 u \, du = \sec u \tan u.$$

As with substitution, we do not have to rely on insight or cleverness to discover such antiderivatives; there is a technique that will often help to uncover the product rule.

Start with the product rule:

$$\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x).$$

We can rewrite this as

$$f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx,$$

and then

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$$

This may not seem particularly useful at first glance, but it turns out that in many cases we have an integral of the form

$$\int f(x)g'(x)\,dx$$

but that

$$\int f'(x)g(x)\,dx$$

is easier. This technique for turning one integral into another is called **integration by parts**, and is usually written in more compact form. If we let $u = f(x)$ and $v = g(x)$ then $du = f'(x)\,dx$ and $dv = g'(x)\,dx$ and

$$\int u\,dv = uv - \int v\,du.$$

To use this technique we need to identify likely candidates for $u = f(x)$ and $dv = g'(x)\,dx$.

Example 3.37: Product of a Linear Function and Logarithm

Evaluate $\int x \ln x \, dx$.

Solution. Let $u = \ln x$ so $du = 1/x\,dx$. Then we must let $dv = x\,dx$ so $v = x^2/2$ and

$$\int x \ln x \, dx = \frac{x^2 \ln x}{2} - \int \frac{x^2}{2}\frac{1}{x}\,dx = \frac{x^2 \ln x}{2} - \int \frac{x}{2}\,dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C.$$

♣

Example 3.38: Product of a Linear Function and Trigonometric Function

Evaluate $\int x \sin x \, dx$.

Solution. Let $u = x$ so $du = dx$. Then we must let $dv = \sin x \, dx$ so $v = -\cos x$ and

$$\int x \sin x \, dx = -x \cos x - \int -\cos x \, dx = -x \cos x + \int \cos x \, dx = -x \cos x + \sin x + C.$$

♣

Example 3.39: Secant Cubed (again)

Evaluate $\int \sec^3 x \, dx$.

Solution. Of course we already know the answer to this, but we needed to be clever to discover it. Here we'll use the new technique to discover the antiderivative. Let $u = \sec x$ and $dv = \sec^2 x \, dx$. Then $du = \sec x \tan x$ and $v = \tan x$ and

$$\begin{aligned} \int \sec^3 x \, dx &= \sec x \tan x - \int \tan^2 x \sec x \, dx \\ &= \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx \\ &= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx. \end{aligned}$$

At first this looks useless—we're right back to $\int \sec^3 x \, dx$. But looking more closely:

$$\begin{aligned} \int \sec^3 x \, dx &= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx \\ \int \sec^3 x \, dx + \int \sec^3 x \, dx &= \sec x \tan x + \int \sec x \, dx \\ 2 \int \sec^3 x \, dx &= \sec x \tan x + \int \sec x \, dx \\ \int \sec^3 x \, dx &= \frac{\sec x \tan x}{2} + \frac{1}{2} \int \sec x \, dx \\ &= \frac{\sec x \tan x}{2} + \frac{\ln|\sec x + \tan x|}{2} + C. \end{aligned}$$

♣

Example 3.40: Product of a Polynomial and Trigonometric Function

Evaluate $\int x^2 \sin x \, dx$.

Solution. Let $u = x^2$, $dv = \sin x \, dx$; then $du = 2x \, dx$ and $v = -\cos x$. Now

$$\int x^2 \sin x \, dx = -x^2 \cos x + \int 2x \cos x \, dx.$$

This is better than the original integral, but we need to do integration by parts again. Let $u = 2x$, $dv = \cos x\,dx$; then $du = 2$ and $v = \sin x$, and

$$\begin{aligned}\int x^2 \sin x\,dx &= -x^2\cos x + \int 2x\cos x\,dx \\ &= -x^2\cos x + 2x\sin x - \int 2\sin x\,dx \\ &= -x^2\cos x + 2x\sin x + 2\cos x + C.\end{aligned}$$

♣

Such repeated use of integration by parts is fairly common, but it can be a bit tedious to accomplish, and it is easy to make errors, especially sign errors involving the subtraction in the formula. There is a nice tabular method to accomplish the calculation that minimizes the chance for error and speeds up the whole process. We illustrate with the previous example. Here is the table:

sign	u	dv
+	x^2	$\sin x$
-	$2x$	$-\cos x$
+	2	$-\sin x$
-	0	$\cos x$

To form this table, we start with u at the top of the second column and repeatedly compute the derivative; starting with dv at the top of the third column, we repeatedly compute the antiderivative. In the first column, we place a "$-$" in every second row. To form the second table we combine the first and second columns by ignoring the boundary; if you do this by hand, you may simply start with two columns and add a "$-$" to every second row.

Alternatively, we can use the following table:

u	dv
x^2	$\sin x$
$-2x$	$-\cos x$
2	$-\sin x$
0	$\cos x$

To compute with this second table we begin at the top. Multiply the first entry in column u by the second entry in column dv to get $-x^2\cos x$, and add this to the integral of the product of the second entry in column u and second entry in column dv. This gives:

$$-x^2\cos x + \int 2x\cos x\,dx,$$

or exactly the result of the first application of integration by parts. Since this integral is not yet easy, we return to the table. Now we multiply twice on the diagonal, $(x^2)(-\cos x)$ and $(-2x)(-\sin x)$ and then once straight across, $(2)(-\sin x)$, and combine these as

$$-x^2\cos x + 2x\sin x - \int 2\sin x\,dx,$$

giving the same result as the second application of integration by parts. While this integral is easy, we may return yet once more to the table. Now multiply three times on the diagonal to get $(x^2)(-\cos x)$, $(-2x)(-\sin x)$, and $(2)(\cos x)$, and once straight across, $(0)(\cos x)$. We combine these as before to get

$$-x^2\cos x + 2x\sin x + 2\cos x + \int 0\,dx = -x^2\cos x + 2x\sin x + 2\cos x + C.$$

Typically we would fill in the table one line at a time, until the "straight across" multiplication gives an easy integral. If we can see that the u column will eventually become zero, we can instead fill in the whole table; computing the products as indicated will then give the entire integral, including the "$+C$", as above.

Exercises for 3.1

Find the antiderivatives.

Exercise 3.1.1 $\int x\cos x\,dx$

Exercise 3.1.2 $\int x^2\cos x\,dx$

Exercise 3.1.3 $\int xe^x\,dx$

Exercise 3.1.4 $\int xe^{x^2}\,dx$

Exercise 3.1.5 $\int \sin^2 x\,dx$

Exercise 3.1.6 $\int \ln x\,dx$

Exercise 3.1.7 $\int x\arctan x\,dx$

Exercise 3.1.8 $\int x^3\sin x\,dx$

Exercise 3.1.9 $\int x^3\cos x\,dx$

Exercise 3.1.10 $\int x\sin^2 x\,dx$

Exercise 3.1.11 $\int x\sin x\cos x\,dx$

Exercise 3.1.12 $\int \arctan(\sqrt{x})\,dx$

Exercise 3.1.13 $\int \sin(\sqrt{x})\,dx$

Exercise 3.1.14 $\int \sec^2 x\csc^2 x\,dx$

Find the antiderivatives.

Exercise 3.1.15 $\int x\sin x\,dx$

Exercise 3.1.16 $\int xe^{-x}\,dx$

Exercise 3.1.17 $\int x^2\sin x\,dx$

Exercise 3.1.18 $\int x^3\sin x\,dx$

Exercise 3.1.19 $\int xe^{x^2}\,dx$

Exercise 3.1.20 $\int x^3e^x\,dx$

Exercise 3.1.21 $\int xe^{-2x}\,dx$

Exercise 3.1.22 $\int e^x\sin x\,dx$

Exercise 3.1.23 $\int e^{2x}\cos x\,dx$

Exercise 3.1.24 $\int e^{2x}\sin(3x)\,dx$

Exercise 3.1.25 $\int e^{5x}\cos(5x)\,dx$

Exercise 3.1.26 $\int \sin x\cos x\,dx$

Exercise 3.1.27 $\int \sin^{-1} x\,dx$

Exercise 3.1.28 $\int \tan^{-1}(2x)\,dx$

Exercise 3.1.29 $\int x\tan^{-1} x\,dx$

Exercise 3.1.30 $\int \sin^{-1} x\,dx$

Exercise 3.1.31 $\int x\ln x\,dx$

Exercise 3.1.32 $\int (x-2)\ln x\,dx$

Exercise 3.1.33 $\int x\ln(x-1)\,dx$

Exercise 3.1.34 $\int x\ln(x^2)\,dx$

Exercise 3.1.35 $\int x^2\ln x\,dx$

Exercise 3.1.36 $\int (\ln x)^2\,dx$

Exercise 3.1.37 $\int (\ln(x+1))^2\,dx$

Exercise 3.1.38 $\int x\sec^2 x\,dx$

Exercise 3.1.39 $\int x\csc^2 x\,dx$

Exercise 3.1.40 $\int x\sqrt{x-2}\,dx$

Exercise 3.1.41 $\int x\sec x\tan x\,dx$

Exercise 3.1.42 $\int x\csc x\cot x\,dx$

Evaluate the indefinite integral after first making a substitution.

Exercise 3.1.43 $\int \sin(\ln x)\,dx$

Exercise 3.1.44 $\int \sin(\sqrt{x})\,dx$

Exercise 3.1.45 $\int \ln(\sqrt{x})\,dx$

Exercise 3.1.46 $\int e^{\sqrt{x}}\,dx$

Exercise 3.1.47 $\int e^{\ln x}\,dx$

Evaluate the definite integral.

Exercise 3.1.48 $\int_0^{\pi} x\sin x\,dx$

Exercise 3.1.49 $\int_{-1}^{1} xe^{-x}\,dx$

Exercise 3.1.50 $\int_{-\pi/4}^{\pi/4} x^2\sin x\,dx$

Exercise 3.1.51 $\int_{-\pi/2}^{\pi/2} x^3\sin x\,dx$

Exercise 3.1.52 $\int_0^{\sqrt{\ln 2}} xe^{x^2}\,dx$

Exercise 3.1.53 $\int_0^{1} x^3e^x\,dx$

Exercise 3.1.54 $\int_1^{2} xe^{-2x}\,dx$

Exercise 3.1.55 $\int_0^{\pi} e^x\sin x\,dx$

Exercise 3.1.56 $\int_{-\pi/2}^{\pi/2} e^{2x}\cos x\,dx$

3.2 Powers of Trigonometric Functions

Functions consisting of powers of the sine and cosine can be integrated by using substitution and trigonometric identities. These can sometimes be tedious, but the technique is straightforward. A similar technique is applicable to powers of secant and tangent (and also cosecant and cotangent, not discussed here).

The trigonometric substitutions we will focus on in this section are summarized in the table below:

Substitution	$u=\sin x$	$u=\cos x$	$u=\tan x$	$u=\sec x$
Derivative	$du=\cos x\,dx$	$du=-\sin x\,dx$	$du=\sec^2 x\,dx$	$du=\sec x\tan x\,dx$

An example will suffice to explain the approach.

Example 3.41: Odd Power of Sine

Evaluate $\int \sin^5 x\,dx$.

Solution. Rewrite the function:

$$\begin{aligned}\int \sin^5 x\,dx &= \int \sin x \sin^4 x\,dx \\ &= \int \sin x(\sin^2 x)^2\,dx \\ &= \int \sin x(1-\cos^2 x)^2\,dx.\end{aligned}$$

Now use $u=\cos x$, $du=-\sin x\,dx$:

$$\begin{aligned}\int \sin x(1-\cos^2 x)^2\,dx &= \int -(1-u^2)^2\,du \\ &= \int -(1-2u^2+u^4)\,du \\ &= \int 1+2u^2-u^4\,du \\ &= -u+\frac{2}{3}u^3-\frac{1}{5}u^5+C \\ &= -\cos x+\frac{2}{3}\cos^3 x-\frac{1}{5}\cos^5 x+C.\end{aligned}$$

♣

Observe that by taking the substitution $u=\cos x$ in the last example, we ended up with an even power of sine from which we can use the formula $\sin^2 x+\cos^2 x=1$ to replace any remaining sines. We then ended up with a polynomial in u in which we could expand and integrate quite easily.

This technique works for products of powers of sine and cosine. We summarize it below.

Products of Sine and Cosine

When evaluating $\int \sin^m x \cos^n x\,dx$:

1. **The power of sine is odd (m odd):**
   (a) Use $u = \cos x$ and $du = -\sin x\,dx$.
   (b) Replace dx using (a), thus cancelling one power of $\sin x$ by the substitution of du, and be left with an even number of sine powers.
   (c) Use $\sin^2 x = 1 - \cos^2 x$ $(= 1 - u^2)$ to replace the leftover sines.

2. **The power of cosine is odd (n odd):**
   (a) Use $u = \sin x$ and $du = \cos x\,dx$.
   (b) Replace dx using (a), thus cancelling one power of $\cos x$ by the substitution of du, and be left with an even number of cosine powers.
   (c) Use $\cos^2 x = 1 - \sin^2 x$ $(= 1 - u^2)$ to replace the leftover cosines.

3. **Both m and n are odd**:
   Use either 1 or 2 (both will work).

4. **Both m and n are even**:
   Use $\cos^2 x = \frac{1}{2}(1 + \cos(2x))$ and/or $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$ to reduce to a form that can be integrated.

NOTE: As m and n get large, multiple steps will be needed.

Example 3.42: Odd Power of Cosine and Even Power of Sine

Evaluate $\int \sin^6 x \cos^5 x\,dx$.

Solution. We will show this solution in two ways. First, we show the solution by rewriting the function to make the substitution more clear. Then, we show the solution using the above method.

Solution 1: Rewrite the function as follows.

$$\begin{aligned}
\int \sin^6 x \cos^5 x\,dx &= \int \sin^6 x \cos^4 x \cos x\,dx \\
&= \int \sin^6 x (\cos^2 x)^2 \cos x\,dx \\
&= \int \sin^6 x (1 - \sin^2 x)^2 \cos x\,dx
\end{aligned}$$

Then use the substitution $u = \sin x$ and $du = \cos x\,dx$, that is, $\boxed{dx} = \boxed{\dfrac{du}{\cos x}}$.

$$\begin{aligned}
\int \sin^6 x (1 - \sin^2 x)^2 \cos x\,dx &= \int u^6 (1 - u^2)^2 du \\
&= \int u^6 - 2u^8 + u^{10}\,du \\
&= \frac{u^7}{7} - \frac{2u^9}{9} + \frac{u^{11}}{11} + C \\
&= \frac{\sin^7 x}{7} - \frac{2\sin^9 x}{9} + \frac{\sin^{11} x}{11} + C
\end{aligned}$$

Solution 2: Since the power of cosine is odd, we use the substitution $u = \sin x$ and $du = \cos x\,dx$, that is, $\boxed{dx} = \boxed{\dfrac{du}{\cos x}}$.

Then $\displaystyle\int \sin^6 x \cos^5 x\,dx$ is equal to:

$$
\begin{aligned}
&= \int u^6 \cos^5 x \boxed{\frac{du}{\cos x}} && \text{Using the substitution}\\
&= \int u^6 \left(\cos^2 x\right)^2 du && \text{Canceling a } \cos x \text{ and rewriting } \cos^4 x\\
&= \int u^6 (1-\sin^2 x)^2\,du && \text{Using trig identity } \cos^2 x = 1-\sin^2 x\\
&= \int u^6 (1-u^2)^2\,du && \text{Writing integral in terms of } u\text{'s}\\
&= \int u^6 - 2u^8 + u^{10}\,du && \text{Expand and collect like terms}\\
&= \frac{u^7}{7} - \frac{2u^9}{9} + \frac{u^{11}}{11} + C && \text{Integrating}\\
&= \frac{\sin^7 x}{7} - \frac{2\sin^9 x}{9} + \frac{\sin^{11} x}{11} + C && \text{Replacing } u \text{ back in terms of } x
\end{aligned}
$$

♣

Example 3.43: Odd Power of Cosine

Evaluate $\displaystyle\int \cos^3 x\,dx$.

Solution. Since the power of cosine is odd, we use the substitution $u = \sin x$ and $du = \cos x\,dx$. This may seem strange at first since we don't have $\sin x$ in the question, but it does work!

$$
\begin{aligned}
\int \cos^3 x \boxed{dx} &= \int \cos^3 x \boxed{\frac{du}{\cos x}} && \text{Using the substitution}\\
&= \int \cos^2 x\,du && \text{Canceling a } \cos x\\
&= \int (1-\sin^2 x)\,du && \text{Using trig identity } \cos^2 x = 1-\sin^2 x\\
&= \int (1-u^2)\,du && \text{Writing integral in terms of } u\text{'s}\\
&= u - \frac{u^3}{3} + C && \text{Integrating}\\
&= \sin x - \frac{\sin^3 x}{3} + C && \text{Replacing } u \text{ back in terms of } x
\end{aligned}
$$

♣

Example 3.44: Product of Even Powers of Sine and Cosine

Evaluate $\displaystyle\int \sin^2 x \cos^2 x\,dx$.

Solution. Use the formulas $\sin^2 x = (1-\cos(2x))/2$ and $\cos^2 x = (1+\cos(2x))/2$ to get:

$$\int \sin^2 x \cos^2 x \, dx = \int \frac{1-\cos(2x)}{2} \cdot \frac{1+\cos(2x)}{2} \, dx.$$

We then have

$$\begin{aligned}
\int \sin^2 x \cos^2 x \, dx &= \int \frac{1-\cos(2x)}{2} \cdot \frac{1+\cos(2x)}{2} \, dx \\
&= \frac{1}{4} \int 1-\cos^2 2x \, dx \\
&= \frac{1}{4}\left(x - \int \cos^2 2x \, dx\right) \\
&= \frac{1}{4}\left(x - \frac{1}{2}\int 1+\cos 4x \, dx\right) \\
&= \frac{1}{4}\left(x - \frac{1}{2}\left(x + \frac{\sin 4x}{4}\right)\right) \\
&= \frac{1}{4}\left(x - \frac{x}{2} - \frac{\sin 4x}{8}\right) + C
\end{aligned}$$

♣

Example 3.45: Even Power of Sine

Evaluate $\int \sin^6 x \, dx.$

Solution. Use $\sin^2 x = (1-\cos(2x))/2$ to rewrite the function:

$$\begin{aligned}
\int \sin^6 x \, dx &= \int (\sin^2 x)^3 \, dx \\
&= \int \frac{(1-\cos 2x)^3}{8} \, dx \\
&= \frac{1}{8} \int 1 - 3\cos 2x + 3\cos^2 2x - \cos^3 2x \, dx.
\end{aligned}$$

Now we have four integrals to evaluate. Ignoring the constant for now:

$$\int 1 \, dx = x$$

and

$$\int -3\cos 2x \, dx = -\frac{3}{2}\sin 2x$$

are easy. The $\cos^3 2x$ integral is like the previous example:

$$\begin{aligned}
\int -\cos^3 2x \, dx &= \int -\cos 2x \cos^2 2x \, dx \\
&= \int -\cos 2x (1-\sin^2 2x) \, dx \\
&= \int -\frac{1}{2}(1-u^2) \, du
\end{aligned}$$

$$= -\frac{1}{2}\left(u - \frac{u^3}{3}\right)$$
$$= -\frac{1}{2}\left(\sin 2x - \frac{\sin^3 2x}{3}\right).$$

And finally we use another trigonometric identity, $\cos^2 x = (1+\cos(2x))/2$:

$$\int 3\cos^2 2x\,dx = 3\int \frac{1+\cos 4x}{2}\,dx = \frac{3}{2}\left(x + \frac{\sin 4x}{4}\right).$$

So at long last we get

$$\int \sin^6 x\,dx = \frac{x}{8} - \frac{3}{16}\sin 2x - \frac{1}{16}\left(\sin 2x - \frac{\sin^3 2x}{3}\right) + \frac{3}{16}\left(x + \frac{\sin 4x}{4}\right) + C.$$

♣

Next, we turn our attention to products of secant and tangent. Some we already know how to do.

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \sec x \tan x\,dx = \sec x + C$$

We can also integrate $\tan x$ quite easily.

Example 3.46: Integrating Tangent

Evaluate $\int \tan x\,dx$.

Solution. Note that $\tan x = \dfrac{\sin x}{\cos x}$ and let $u = \cos x$, so that $du = -\sin x\,dx$.

$$\begin{aligned}
\int \tan x\,dx &= \int \frac{\sin x}{\cos x}\boxed{dx} && \text{Rewriting } \tan x\\
&= \int \frac{\sin x}{u}\boxed{\frac{du}{-\sin x}} && \text{Using the substitution}\\
&= -\int \frac{1}{u}\,du && \text{Cancelling and pulling the } -1 \text{ out}\\
&= -\ln|u| + C && \text{Using formula } \int \frac{1}{u}\,dx = \ln|u| + C\\
&= -\ln|\cos x| + C && \text{Replacing } u \text{ back in terms of } x\\
&= \ln|\sec x| + C && \text{Using log properties and } \sec x = 1/\cos x
\end{aligned}$$

♣

Let's take a moment to realize this result! A common mistake is to believe that $\int \tan x\,dx$ is $\sec^2(x) + C$ – this is *not* true.

Example 3.47: Integrating Tangent Squared

Evaluate $\int \tan^2 x\,dx$.

Solution. Note that $\tan^2 x = \sec^2 x - 1$.

$$\begin{aligned}\int \tan^2 x\,dx &= \int \sec^2 x - 1\,dx && \text{Rewriting } \tan x\\ &= \tan x - x + C && \text{Since } \int \sec^2 x\,dx = \tan x + C\end{aligned}$$

♣

In problems with tangent and secant, two integrals come up frequently:

$$\int \sec^3 x\,dx \qquad \text{and} \qquad \int \sec x\,dx.$$

Both have relatively nice expressions but they are a bit tricky to discover.

First we do $\int \sec x\,dx$, which we will need to compute $\int \sec^3 x\,dx$.

Example 3.48: Integral of Secant

Evaluate $\int \sec x\,dx$.

Solution.

$$\begin{aligned}\int \sec x\,dx &= \int \sec x\,\frac{\sec x + \tan x}{\sec x + \tan x}\,dx\\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx.\end{aligned}$$

Now let $u = \sec x + \tan x$, $du = \sec x \tan x + \sec^2 x\,dx$, exactly the numerator of the function we are integrating. Thus

$$\begin{aligned}\int \sec x\,dx &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx\\ &= \int \frac{1}{u}\,du = \ln|u| + C\\ &= \ln|\sec x + \tan x| + C.\end{aligned}$$

♣

Now we compute the integral $\int \sec^3 x\,dx$.

Example 3.49: Integral of Secant Cubed

Evaluate $\int \sec^3 x\,dx$.

Solution.

$$\begin{aligned}\sec^3 x &= \frac{\sec^3 x}{2} + \frac{\sec^3 x}{2} = \frac{\sec^3 x}{2} + \frac{(\tan^2 x + 1)\sec x}{2}\\ &= \frac{\sec^3 x}{2} + \frac{\sec x \tan^2 x}{2} + \frac{\sec x}{2}\end{aligned}$$

$$= \quad \frac{\sec^3 x + \sec x \tan^2 x}{2} + \frac{\sec x}{2}.$$

We already know how to integrate $\sec x$, so we just need the first quotient. This is “simply” a matter of recognizing the product rule in action:

$$\int \sec^3 x + \sec x \tan^2 x\,dx = \sec x \tan x.$$

So putting these together we get

$$\int \sec^3 x\,dx = \frac{\sec x \tan x}{2} + \frac{\ln|\sec x + \tan x|}{2} + C,$$

♣

For products of secant and tangent it is best to use the following guidelines.

Products of Secant and Tangent

When evaluating $\int \sec^m x \tan^n x\,dx$:

1. **The power of secant is even (m even):**
   (a) Use $u = \tan x$ and $du = \sec^2 x\,dx$.
   (b) Cancel $\sec^2 x$ by the substitution of dx, and be left with an even number of secants.
   (c) Use $\sec^2 x = 1 + \tan^2 x$ $(= 1 + u^2)$ to replace the leftover secants.

2. **The power of tangent is odd (n odd):**
   (a) Use $u = \sec x$ and $du = \sec x \tan x\,dx$.
   (b) Cancel one $\sec x$ and one $\tan x$ by the substitution of dx.
   The number of remaining tangents is even.
   (c) Use $\tan^2 x = \sec^2 x - 1$ $(= u^2 - 1)$ to replace the leftover tangents.

3. **m is even or n is odd**:
   Use either 1 or 2 (both will work).

4. **The power of secant is odd and the power of tangent is even**:
   No guidelines. Remember that $\int \sec x\,dx$ and $\int \sec^3 x\,dx$ can usually be looked up.

Example 3.50: Even Power of Secant

Evaluate $\int \sec^6 x \tan^6 x\,dx$.

Solution. Since the power of secant is even, we ue $u = \tan x$, so that $du = \sec^2 x\,dx$.

$$\begin{aligned}
\int \sec^6 x \tan^6 x\,dx &= \int \sec^6 x\,(u^6) \boxed{\frac{du}{\sec^2 x}} && \text{Using the substitution} \\
&= \int \sec^4 x (u^6)\,du && \text{Cancelling a } \sec^2 x \\
&= \int (\sec^2 x)^2 (u^6)\,du && \text{Rewriting } \sec^4 x \\
&= \int (1+\tan^2 x)^2 (u^6)\,du && \text{Using } \sec^2 x = 1+\tan^2 x \\
&= \int (1+u^2)^2 (u^6)\,du && \text{Using the substitution}
\end{aligned}$$

To integrate this product the easiest method is expand it into a polynomial and integrate term-by-term.

$$\begin{aligned}
\int \sec^6 x \tan^6 x\,dx &= \int (u^6 + 2u^8 + u^{10})\,du && \text{Expanding} \\
&= \frac{u^7}{7} + \frac{2u^9}{9} + \frac{u^{11}}{11} + C && \text{Integrating} \\
&= \frac{\tan^7 x}{7} + \frac{2\tan^9 x}{9} + \frac{\tan^{11} x}{11} + C && \text{Rewriting in terms of } x
\end{aligned}$$

♣

Example 3.51: Odd Power of Tangent

Evaluate $\int \sec^5 x \tan x\,dx$.

Solution. Since the power of tangent is odd, we use $u = \sec x$, so that $du = \sec x \tan x\,dx$. Then we have:

$$\begin{aligned}
\int \sec^5 x \tan x\,dx &= \int \sec^5 x \tan x \boxed{\frac{du}{\sec x \tan x}} && \text{Substituting } dx \text{ first} \\
&= \int \sec^4 x\,du && \text{Cancelling} \\
&= \int u^4\,du && \text{Using the substitution} \\
&= \frac{u^5}{5} + C && \text{Integrating} \\
&= \frac{\sec^5 x}{5} + C && \text{Rewriting in terms of } x
\end{aligned}$$

♣

Example 3.52: Odd Power of Secant and Even Power of Tangent

Evaluate $\int \sec x \tan^2 x\,dx$.

Solution. The guidelines don't help us in this scenario. However, since $\tan^2 x = \sec^2 x - 1$, we have

$$\begin{aligned}
\int \sec x \tan^2 x\,dx &= \int \sec x(\sec^2 x - 1)\,dx \\
&= \int (\sec^3 x - \sec x)\,dx \\
&= \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) - \ln|\sec x + \tan x| + C \\
&= \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| - \ln|\sec x + \tan x| + C \\
&= \frac{1}{2}\sec x \tan x - \frac{1}{2}\ln|\sec x + \tan x| + C
\end{aligned}$$

♣

Exercises for 3.2

Find the antiderivatives.

Exercise 3.2.1 $\int \sin^2 x\,dx$

Exercise 3.2.2 $\int \sin^3 x\,dx$

Exercise 3.2.3 $\int \sin^4 x\,dx$

Exercise 3.2.4 $\int \cos^2 x \sin^3 x\,dx$

Exercise 3.2.5 $\int \cos^3 x\,dx$

Exercise 3.2.6 $\int \cos^3 x \sin^2 x\,dx$

Exercise 3.2.7 $\int \sin x(\cos x)^{3/2}\,dx$

Exercise 3.2.8 $\int \sec^2 x \csc^2 x\,dx$

Exercise 3.2.9 $\int \tan^3 x \sec x\,dx$

Exercise 3.2.10 $\int \left(\frac{1}{\csc x} + \frac{1}{\sec x}\right) dx$

Exercise 3.2.11 $\int \frac{\cos^2 x + \cos x + 1}{\cos^3 x}\,dx$

Exercise 3.2.12 $\int x \sec^2(x^2) \tan^4(x^2)\,dx$

Evaluate the given indefinite integrals.

Exercise 3.2.13 $\int \sin x \cos^4 x\,dx$

Exercise 3.2.14 $\int \sin^3 x \cos x\,dx$

Exercise 3.2.15 $\int \sin^3 x \cos^2 x\,dx$

Exercise 3.2.16 $\int \sin^3 x \cos^3 x\,dx$

Exercise 3.2.17 $\int \sin^6 x \cos^5 x\,dx$

Exercise 3.2.18 $\int \sin^2 x \cos^7 x\,dx$

Exercise 3.2.19 $\int \sin^2 x \cos^2 x \, dx$

Exercise 3.2.20 $\int \sin(5x)\cos(3x) \, dx$

Exercise 3.2.21 $\int \sin(x)\cos(2x) \, dx$

Exercise 3.2.22 $\int \sin(3x)\sin(7x) \, dx$

Exercise 3.2.23 $\int \sin(\pi x)\sin(2\pi x) \, dx$

Exercise 3.2.24 $\int \cos(x)\cos(2x) \, dx$

Exercise 3.2.25 $\int \cos\left(\frac{\pi}{2}x\right)\cos(\pi x) \, dx$

Exercise 3.2.26 $\int \tan^4 x \sec^2 x \, dx$

Exercise 3.2.27 $\int \tan^2 x \sec^4 x \, dx$

Exercise 3.2.28 $\int \tan^3 x \sec^4 x \, dx$

Exercise 3.2.29 $\int \tan^3 x \sec^2 x \, dx$

Exercise 3.2.30 $\int \tan^3 x \sec^3 x \, dx$

Exercise 3.2.31 $\int \tan^5 x \sec^5 x \, dx$

Exercise 3.2.32 $\int \tan^4 x \, dx$

Exercise 3.2.33 $\int \sec^5 x \, dx$

Exercise 3.2.34 $\int \tan^2 x \sec x \, dx$

Exercise 3.2.35 $\int \tan^2 x \sec^3 x \, dx$

3.3 Trigonometric Substitutions

So far we have seen that it sometimes helps to replace a subexpression of a function by a single variable. Occasionally it can help to replace the original variable by something more complicated. This seems like a "reverse" substitution, but it is really no different in principle than ordinary substitution.

Example 3.53: Sine Subsitution

Evaluate $\int \sqrt{1-x^2}\, dx$.

Solution. Let $x = \sin u$ so $dx = \cos u \, du$. Then

$$\int \sqrt{1-x^2}\, dx = \int \sqrt{1-\sin^2 u}\cos u \, du = \int \sqrt{\cos^2 u}\cos u \, du.$$

We would like to replace $\sqrt{\cos^2 u}$ by $\cos u$, but this is valid only if $\cos u$ is positive, since $\sqrt{\cos^2 u}$ is positive. Consider again the substitution $x = \sin u$. We could just as well think of this as $u = \arcsin x$. If we do, then by the definition of the arcsine, $-\pi/2 \le u \le \pi/2$, so $\cos u \ge 0$. Then we continue:

$$\begin{aligned}
\int \sqrt{\cos^2 u}\cos u\,du &= \int \cos^2 u\,du \\
&= \int \frac{1+\cos 2u}{2}\,du = \frac{u}{2}+\frac{\sin 2u}{4}+C \\
&= \frac{\arcsin x}{2}+\frac{\sin(2\arcsin x)}{4}+C.
\end{aligned}$$

This is a perfectly good answer, though the term $\sin(2\arcsin x)$ is a bit unpleasant. It is possible to simplify this. Using the identity $\sin 2x = 2\sin x\cos x$, we can write $\sin 2u = 2\sin u\cos u = 2\sin(\arcsin x)\sqrt{1-\sin^2 u} = 2x\sqrt{1-\sin^2(\arcsin x)} = 2x\sqrt{1-x^2}$. Then the full antiderivative is

$$\frac{\arcsin x}{2}+\frac{2x\sqrt{1-x^2}}{4} = \frac{\arcsin x}{2}+\frac{x\sqrt{1-x^2}}{2}+C.$$

♣

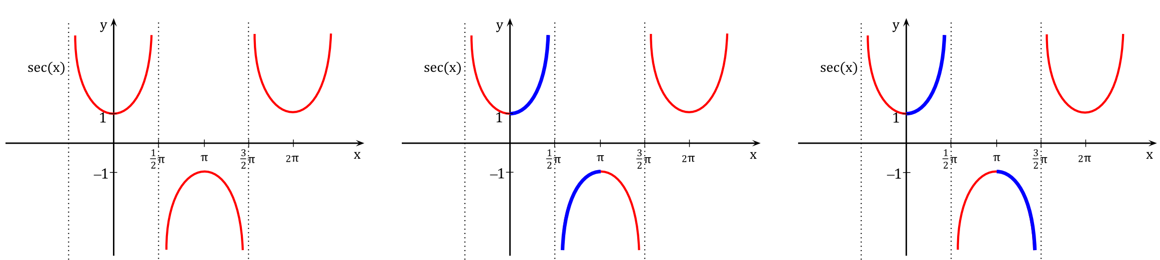
This type of substitution is usually indicated when the function you wish to integrate contains a polynomial expression that might allow you to use the fundamental identity $\sin^2 x+\cos^2 x = 1$ in one of three forms:

$$\cos^2 x = 1-\sin^2 x \qquad \sec^2 x = 1+\tan^2 x \qquad \tan^2 x = \sec^2 x - 1.$$

If your function contains $1-x^2$, as in the example above, try $x=\sin u$; if it contains $1+x^2$ try $x=\tan u$; and if it contains x^2-1, try $x=\sec u$. Sometimes you will need to try something a bit different to handle constants other than one which we will describe below. First we discuss inverse substitutions.

In a **traditional** substitution we let $u=u(x)$, i.e., our new variable is defined in terms of x. In an **inverse** substitution we let $x=g(u)$, i.e., we assume x can be written in terms of u. We cannot do this arbitrarily since we do **NOT** get to "choose" x. For example, an inverse substitution of $x=1$ will give an obviously wrong answer. However, when $x=g(u)$ is an invertible function, then we are really doing a u-substitution with $u=g^{-1}(x)$. Now the substitution rule applies.

Sometimes with inverse substitutions involving trig functions we use θ instead of u. Thus, we would take $x=\sin\theta$ instead of $x=\sin u$. However, as we discussed above, we would like our inverse substitution $x=g(u)$ to be a one-to-one function, and $x=\sin u$ is not one-to-one. We can overcome this issue by using the restricted trigonometric functions. The three common trigonometric substitutions are the restricted sine, restricted tangent and restricted secant. Thus, for sine we use the domain $[-\pi/2,\ \pi/2]$ and for tangent we use $(-\pi/2,\ \pi/2)$. Depending on the convention chosen, the restricted secant function is usually defined in one of two ways.



One convention is to restrict secant to the region $[0,\ \pi/2)\cup(\pi/2,\pi]$ as shown in the middle graph. The other convention is to use $[0,\ \pi/2)\cup[\pi,\ 3\pi/2)$ as shown in the right graph. Both choices give a one-to-one restricted secant function and no universal convention has been adopted. To make the analysis in this section less cumbersome, we will use the domain $[0,\ \pi/2)\cup[\pi,\ 3\pi/2)$ for the restricted secant function. Then $\sec^{-1} x$ is defined to be the inverse of this restricted secant function.

Typically trigonometric substitions are used for problems that involve radical expressions. The table below outlines when each substitution is typically used along with their intervals of validity.

Expression	**Substitution**	**Validity**
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$\theta \in [-\pi/2,\ \pi/2]$
$\sqrt{a^2+x^2}$ or a^2+x^2	$x = a\tan\theta$	$\theta \in (-\pi/2,\ \pi/2)$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$\theta \in [0,\ \pi/2)\cup[\pi,\ 3\pi/2)$

All three substitutions are one-to-one on the listed intervals. When dealing with radicals we often end up with absolute values since

$$\sqrt{z^2} = |z|.$$

For each of the three trigonometric substitions above we will verify that we can ignore the absolute value in each case when encountering a radical.

For $x = a\sin\theta$, the expression $\sqrt{a^2-x^2}$ becomes

$$\sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2\theta} = \sqrt{a^2(1-\sin^2\theta)} = a\sqrt{\cos^2\theta} = a|\cos\theta| = a\cos\theta$$

This is because $\cos\theta \geq 0$ when $\theta \in [-\pi/2,\ \pi/2]$. For $x = a\tan\theta$, the expression $\sqrt{a^2+x^2}$ becomes

$$\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2\theta} = \sqrt{a^2(1+\tan^2\theta)} = a\sqrt{\sec^2\theta} = a|\sec\theta| = a\sec\theta$$

This is because $\sec\theta > 0$ when $\theta \in (-\pi/2,\ \pi/2)$.

Finally, for $x = a\sec\theta$, the expression $\sqrt{x^2-a^2}$ becomes

$$\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2(\sec^2\theta-1)} = a\sqrt{\tan^2\theta} = a|\tan\theta| = a\tan\theta$$

This is because $\tan\theta \geq 0$ when $\theta \in [0,\ \pi/2)\cup[\pi,\ 3\pi/2)$.

Thus, when using an appropriate trigonometric substitution we can usually ignore the absolute value. After integrating, we typically get an answer in terms of θ (or u) and need to convert back to x's. To do so, we use the two guidelines below:

- For trig functions containing θ, use a triangle to convert to x's.
- For θ by itself, use the inverse trig function.

All pieces needed for such a trigonometric substitution can be summarized as follows:

Expression	Substitution	Differential	Identity	Inverse of Substitution
$\sqrt{a^2-x^2}$	$x=a\sin\theta$	$dx=a\cos\theta d\theta$	$\sqrt{a^2-x^2}=a\cos\theta$	$\theta=\sin^{-1}\left(\frac{x}{a}\right)$
$\sqrt{a^2+x^2}$ or a^2+x^2	$x=a\tan\theta$	$dx=a\sec^2\theta d\theta$	$\sqrt{a^2+x^2}=a\sec\theta$	$\theta=\tan^{-1}\left(\frac{x}{a}\right)$
$\sqrt{x^2-a^2}$	$x=a\sec\theta$	$dx=a\sec\theta\tan\theta d\theta$	$\sqrt{x^2-a^2}=a\tan\theta$	$\theta=\sec^{-1}\left(\frac{x}{a}\right)$

To emphasize the technique, we redo the computation for $\int\sqrt{1-x^2}\,dx$.

Example 3.54: Sine Subsitution

Evaluate $\int\sqrt{1-x^2}\,dx$.

Solution. Since $\sqrt{1-x^2}$ appears in the integrand we try the trigonometric substitution $x=\sin\theta$. (Here we are using the restricted sine function with $\theta\in[-\pi/2,\ \pi/2]$ but typically omit this detail when writing out the solution.) Then $\boxed{dx}=\boxed{\cos\theta\,d\theta}$.

$$\begin{aligned}
\int\sqrt{1-x^2}\,\boxed{dx} &= \int\sqrt{1-\sin^2\theta}\,\boxed{\cos\theta\,d\theta} && \text{Using our (inverse) substitution}\\
&= \int\sqrt{\cos^2\theta}\cos\theta\,d\theta && \text{Since } \sin^2\theta+\cos^2\theta=1\\
&= \int|\cos\theta|\cdot\cos\theta\,d\theta && \text{Since } \sqrt{\cos^2\theta}=|\cos\theta|\\
&= \int\cos^2\theta\,d\theta && \text{Since for } \theta\in[-\tfrac{\pi}{2},\tfrac{\pi}{2}] \text{ we have } \cos\theta\geq 0.
\end{aligned}$$

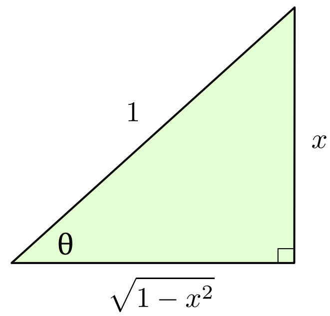
Often we omit the step containing the absolute value by our discussion above. Now, to integrate a power of cosine we use the guidelines for products of sine and cosine and make use of the identity

$$\cos^2\theta=\frac{1}{2}(1+\cos(2\theta)).$$

Our integral then becomes

$$\int\sqrt{1-x^2}\,dx=\frac{1}{2}\int(1+\cos(2\theta))\,d\theta=\frac{\theta}{2}+\frac{\sin(2\theta)}{4}+C$$

To write the answer back in terms of x we use a right triangle. Since $\sin\theta = x/1$ we have the triangle:



The triangle gives $\sin\theta$, $\cos\theta$, $\tan\theta$, but have a $\sin(2\theta)$. Thus, we use an identity to write

$$\sin(2\theta) = 2\sin\theta\cos\theta = 2\left(\frac{x}{1}\right)\left(\frac{\sqrt{1-x^2}}{1}\right)$$

For θ by itself we use $\theta = \sin^{-1}x$. Thus, the integral is

$$\int \sqrt{1-x^2}\,dx = \frac{\sin^{-1}x}{2} + \frac{x\sqrt{1-x^2}}{2} + C$$

♣

Example 3.55: Secant Subsitution

Evaluate $\displaystyle\int \frac{\sqrt{25x^2-4}}{x}\,dx$.

Solution. We do not have $\sqrt{x^2-a^2}$ because of the 25, but if we factor 25 out we get:

$$\int \frac{\sqrt{25(x^2-(4/25))}}{x}\,dx = \int 5\frac{\sqrt{x^2-(4/25)}}{x}\,dx.$$

Now, $a = 2/5$, so let $x = \frac{2}{5}\sec\theta$. Alternatively, we can think of the integral as being:

$$\int \frac{\sqrt{(5x)^2-4}}{x}\,dx$$

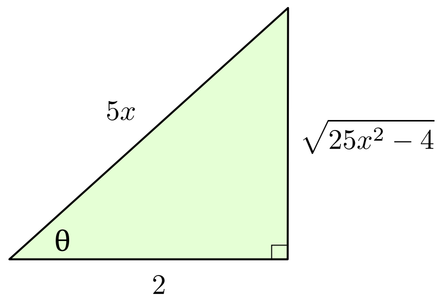
Then we could let $u = 5x$ followed by $u = 2\sec\theta$, etc. Or equivalently, we can avoid a u-substitution by letting $5x = 2\sec\theta$. In either case we are using the trigonometric substitution $x = \frac{2}{5}\sec\theta$, but do use the method that makes the most sense to you! As $x = \frac{2}{5}\sec\theta$ we have $\boxed{dx} = \boxed{\frac{2}{5}\sec\theta\tan\theta\,d\theta}$.

$$\begin{aligned}
\int \frac{\sqrt{25x^2-4}}{x}\boxed{dx} &= \int \frac{\sqrt{25\frac{4\sec^2\theta}{25}-4}}{\frac{2}{5}\sec\theta}\boxed{\frac{2}{5}\sec\theta\tan\theta\,d\theta} && \text{Using the substitution}\\
&= \int \sqrt{4(\sec^2\theta-1)}\cdot\tan\theta\,d\theta && \text{Cancelling}\\
&= 2\int \sqrt{\tan^2\theta}\cdot\tan\theta\,d\theta && \text{Using } \tan^2\theta+1=\sec^2\theta\\
&= 2\int \tan^2\theta\,d\theta && \text{Simplifying}\\
&= 2\int (\sec^2\theta-1)\,d\theta && \text{Using } \tan^2\theta+1=\sec^2\theta\\
&= 2(\tan\theta-\theta)+C && \text{Since } \int \sec^2\theta\,d\theta = \tan\theta + C
\end{aligned}$$

For $\tan\theta$, we use a right triangle.

$$x = \frac{2}{5}\sec\theta \qquad \rightarrow \qquad x = \frac{2}{5}\frac{1}{\cos\theta} \qquad \rightarrow \qquad \cos\theta = \frac{2}{5x}$$

Using SOH CAH TOA, the triangle is then



For θ by itself, we use $\theta = \sec^{-1}(5x/2)$. Thus,

$$\int \frac{\sqrt{25x^2-4}}{x}\,dx = 2\left(\frac{\sqrt{25x^2-4}}{2} - \sec^{-1}\left(\frac{5x}{2}\right)\right) + C$$

♣

In the context of the previous example, some resources give alternate guidelines when choosing a trigonometric substitution.

$$\sqrt{a^2-b^2x^2} \quad \rightarrow \quad x = \frac{a}{b}\sin\theta$$

$$\sqrt{b^2x^2+a^2} \text{ or } (b^2x^2+a^2) \quad \rightarrow \quad x = \frac{a}{b}\tan\theta$$

$$\sqrt{b^2x^2-a^2} \quad \rightarrow \quad x = \frac{a}{b}\sec\theta$$

We next look at a tangent substitution.

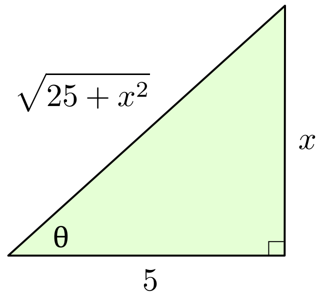
Example 3.56: Tangent Substitution

Evaluate $\int \frac{1}{\sqrt{25+x^2}}\,dx$.

Solution. Let $x = 5\tan\theta$ so that $\boxed{dx} = \boxed{5\sec^2\theta\,d\theta}$.

$$\begin{aligned}
\int \frac{1}{\sqrt{25+x^2}}\boxed{dx} &= \int \frac{1}{\sqrt{25+25\tan^2\theta}}\boxed{5\sec^2\theta\,d\theta} && \text{Using our substitution} \\
&= \int \frac{1}{\sqrt{25(1+\tan^2\theta)}}\cdot 5\sec^2\theta\,d\theta && \text{Factor out 25} \\
&= \int \frac{1}{5\sqrt{sec^2\theta}}\cdot 5\sec^2\theta\,d\theta && \text{Using } \tan^2\theta+1=\sec^2\theta \\
&= \int \sec\theta\,d\theta && \text{Simplifying} \\
&= \ln|\sec\theta+\tan\theta|+C && \text{By } \int \sec\theta\,dx = \ln|\sec\theta+\tan\theta|+C
\end{aligned}$$

Since $\tan\theta = x/5$, we draw a triangle:



Then

$$\sec\theta = \frac{1}{\cos\theta} = \frac{\sqrt{25+x^2}}{5}.$$

Therefore, the integral is

$$\int \frac{1}{\sqrt{25+x^2}}\,dx = \ln\left|\frac{\sqrt{25+x^2}}{5} + \frac{x}{5}\right| + C$$

♣

In the next example, we will use the technique of completing the square in order to rewrite the integrand.

Example 3.57: Completing the Square

Evaluate $\int \frac{x}{\sqrt{3-2x-x^2}}\,dx.$

Solution. First, complete the square to write

$$3-2x-x^2 = 4-(x+1)^2$$

Now, we may let $u = x+1$ so that $du = dx$ (note that $x = u-1$) to get:

$$\int \frac{x}{\sqrt{4-(x+1)^2}}\,dx = \int \frac{u-1}{\sqrt{4-u^2}}\,du$$

Let $u = 2\sin\theta$ giving $du = 2\cos\theta\,d\theta$:

$$\int \frac{u-1}{\sqrt{4-u^2}}\,du = \int \frac{2\sin\theta - 1}{2\cos\theta}\cdot 2\cos\theta\,d\theta = \int (2\sin\theta - 1)\,d\theta$$

Integrating and using a triangle we get:

$$\begin{aligned}
\int \frac{x}{\sqrt{3-2x-x^2}} &= -2\cos\theta - \theta + C \\
&= -\sqrt{4-u^2} - \sin^{-1}\left(\frac{u}{2}\right) + C \\
&= -\sqrt{3-2x-x^2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C
\end{aligned}$$

Note that in this problem we could have skipped the u-substitution if instead we let $x+1 = 2\sin\theta$. (For the triangle we would then use $\sin\theta = \frac{x+1}{2}$.)

♣

Exercises for 3.3

Exercise 3.3.1 $\int \sqrt{x^2-1}\,dx$

Exercise 3.3.2 $\int \sqrt{9+4x^2}\,dx$

Exercise 3.3.3 $\int x\sqrt{1-x^2}\,dx$

Exercise 3.3.4 $\int x^2\sqrt{1-x^2}\,dx$

Exercise 3.3.5 $\int \frac{1}{\sqrt{1+x^2}}\,dx$

Exercise 3.3.6 $\int \sqrt{x^2+2x}\,dx$

Exercise 3.3.7 $\int \frac{1}{x^2(1+x^2)}\,dx$

Exercise 3.3.8 $\int \frac{x^2}{\sqrt{4-x^2}}\,dx$

Exercise 3.3.9 $\int \frac{\sqrt{x}}{\sqrt{1-x}}\,dx$

Exercise 3.3.10 $\int \frac{x^3}{\sqrt{4x^2-1}}\,dx$

Apply Trigonometric Substitution to evaluate the indefinite integrals.

Exercise 3.3.11 $\int \sqrt{x^2+1}\,dx$

Exercise 3.3.12 $\int \sqrt{x^2+4}\,dx$

Exercise 3.3.13 $\int \sqrt{1-x^2}\,dx$

Exercise 3.3.14 $\int \sqrt{9-x^2}\,dx$

Exercise 3.3.15 $\int \sqrt{x^2-1}\,dx$

Exercise 3.3.16 $\int \sqrt{x^2-16}\,dx$

Exercise 3.3.17 $\int \sqrt{4x^2+1}\,dx$

Exercise 3.3.18 $\int \sqrt{1-9x^2}\,dx$

Exercise 3.3.19 $\int \sqrt{16x^2-1}\,dx$

Exercise 3.3.20 $\int \frac{8}{\sqrt{x^2+2}}\,dx$

Exercise 3.3.21 $\int \frac{3}{\sqrt{7-x^2}}\,dx$

Exercise 3.3.22 $\int \frac{5}{\sqrt{x^2-8}}\,dx$

3.4 Rational Functions

A **rational function** is a fraction with polynomials in the numerator and denominator. For example,

$$\frac{x^3}{x^2+x-6}, \qquad \frac{1}{(x-3)^2}, \qquad \frac{x^2+1}{x^2-1},$$

are all rational functions of x. There is a general technique called "partial fractions" that, in principle, allows us to integrate any rational function. The algebraic steps in the technique are rather cumbersome if the polynomial in the denominator has degree more than 2, and the technique requires that we factor the denominator, something that is not always possible. However, in practice one does not often run across rational functions with high degree polynomials in the denominator for which one has to find the antiderivative function. So we shall explain how to find the antiderivative of a rational function only when the denominator is a quadratic polynomial ax^2+bx+c.

We should mention a special type of rational function that we already know how to integrate: If the denominator has the form $(ax+b)^n$, the substitution $u=ax+b$ will always work. The denominator becomes u^n, and each x in the numerator is replaced by $(u-b)/a$, and $dx = du/a$. While it may be tedious to complete the integration if the numerator has high degree, it is merely a matter of algebra.

Example 3.58: Substitution and Splitting Up a Fraction

Find $\displaystyle\int \frac{x^3}{(3-2x)^5}\,dx.$

Solution. Using the substitution $u=3-2x$ we get

$$\begin{aligned}
\int \frac{x^3}{(3-2x)^5}\,dx &= \frac{1}{-2}\int \frac{\left(\frac{u-3}{-2}\right)^3}{u^5}\,du = \frac{1}{16}\int \frac{u^3-9u^2+27u-27}{u^5}\,du \\
&= \frac{1}{16}\int u^{-2}-9u^{-3}+27u^{-4}-27u^{-5}\,du \\
&= \frac{1}{16}\left(\frac{u^{-1}}{-1}-\frac{9u^{-2}}{-2}+\frac{27u^{-3}}{-3}-\frac{27u^{-4}}{-4}\right)+C \\
&= \frac{1}{16}\left(\frac{(3-2x)^{-1}}{-1}-\frac{9(3-2x)^{-2}}{-2}+\frac{27(3-2x)^{-3}}{-3}-\frac{27(3-2x)^{-4}}{-4}\right)+C \\
&= -\frac{1}{16(3-2x)}+\frac{9}{32(3-2x)^2}-\frac{9}{16(3-2x)^3}+\frac{27}{64(3-2x)^4}+C
\end{aligned}$$

♣

We now proceed to the case in which the denominator is a quadratic polynomial. We can always factor out the coefficient of x^2 and put it outside the integral, so we can assume that the denominator has the form x^2+bx+c. There are three possible cases, depending on how the quadratic factors: either $x^2+bx+c=(x-r)(x-s)$, $x^2+bx+c=(x-r)^2$, or it doesn't factor. We can use the quadratic formula to decide which of these we have, and to factor the quadratic if it is possible.

Example 3.59: Factoring a Quadratic

Determine whether x^2+x+1 *factors, and factor it if possible.*

Solution. The quadratic formula tells us that $x^2+x+1=0$ when

$$x=\frac{-1\pm\sqrt{1-4}}{2}.$$

Since there is no square root of -3, this quadratic does not factor. ♣

Example 3.60: Factoring a Quadratic with Real Roots

Determine whether x^2-x-1 factors, and factor it if possible.

Solution. The quadratic formula tells us that $x^2-x-1=0$ when

$$x=\frac{1\pm\sqrt{1+4}}{2}=\frac{1\pm\sqrt{5}}{2}.$$

Therefore

$$x^2-x-1=\left(x-\frac{1+\sqrt{5}}{2}\right)\left(x-\frac{1-\sqrt{5}}{2}\right).$$

♣

If $x^2+bx+c=(x-r)^2$ then we have the special case we have already seen, that can be handled with a substitution. The other two cases require different approaches.

If $x^2+bx+c=(x-r)(x-s)$, we have an integral of the form

$$\int\frac{p(x)}{(x-r)(x-s)}\,dx$$

where $p(x)$ is a polynomial. The first step is to make sure that $p(x)$ has degree less than 2.

Example 3.61:

Rewrite

$$\int\frac{x^3}{(x-2)(x+3)}\,dx$$

in terms of an integral with a numerator that has degree less than 2.

Solution. To do this we use long division of polynomials to discover that

$$\frac{x^3}{(x-2)(x+3)}=\frac{x^3}{x^2+x-6}=x-1+\frac{7x-6}{x^2+x-6}=x-1+\frac{7x-6}{(x-2)(x+3)}.$$

Then

$$\int\frac{x^3}{(x-2)(x+3)}\,dx=\int x-1\,dx+\int\frac{7x-6}{(x-2)(x+3)}\,dx.$$

The first integral is easy, so only the second requires some work. ♣

Now consider the following simple algebra of fractions:

$$\frac{A}{x-r}+\frac{B}{x-s}=\frac{A(x-s)+B(x-r)}{(x-r)(x-s)}=\frac{(A+B)x-As-Br}{(x-r)(x-s)}.$$

That is, adding two fractions with constant numerator and denominators $(x-r)$ and $(x-s)$ produces a fraction with denominator $(x-r)(x-s)$ and a polynomial of degree less than 2 for the numerator. We want to reverse this process: Starting with a single fraction, we want to write it as a sum of two simpler fractions. An example should make it clear how to proceed.

Example 3.62: Partial Fraction Decomposition

Evaluate $\displaystyle\int \frac{x^3}{(x-2)(x+3)}\,dx$.

Solution. We start by writing $\displaystyle\frac{7x-6}{(x-2)(x+3)}$ as the sum of two fractions. We want to end up with

$$\frac{7x-6}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3}.$$

If we go ahead and add the fractions on the right hand side, we seek a common denominator, and get:

$$\frac{7x-6}{(x-2)(x+3)} = \frac{(A+B)x+3A-2B}{(x-2)(x+3)}.$$

So all we need to do is find A and B so that $7x-6=(A+B)x+3A-2B$, which is to say, we need $7=A+B$ and $-6=3A-2B$. This is a problem you've seen before: Solve a system of two equations in two unknowns. There are many ways to proceed; here's one: If $7=A+B$ then $B=7-A$ and so $-6=3A-2B=3A-2(7-A)=3A-14+2A=5A-14$. This is easy to solve for A: $A=8/5$, and then $B=7-A=7-8/5=27/5$. Thus

$$\int \frac{7x-6}{(x-2)(x+3)}\,dx = \int \frac{8}{5}\frac{1}{x-2} + \frac{27}{5}\frac{1}{x+3}\,dx = \frac{8}{5}\ln|x-2| + \frac{27}{5}\ln|x+3| + C.$$

The answer to the original problem is now

$$\begin{aligned}\int \frac{x^3}{(x-2)(x+3)}\,dx &= \int x-1\,dx + \int \frac{7x-6}{(x-2)(x+3)}\,dx \\ &= \frac{x^2}{2} - x + \frac{8}{5}\ln|x-2| + \frac{27}{5}\ln|x+3| + C.\end{aligned}$$

♣

Now suppose that x^2+bx+c doesn't factor. Again we can use long division to ensure that the numerator has degree less than 2, then we complete the square.

Example 3.63: Denominator Does Not Factor

Evaluate $\displaystyle\int \frac{x+1}{x^2+4x+8}\,dx$.

Solution. The quadratic denominator does not factor. We could complete the square and use a trigonometric substitution, but it is simpler to rearrange the integrand:

$$\int \frac{x+1}{x^2+4x+8}\,dx = \int \frac{x+2}{x^2+4x+8}\,dx - \int \frac{1}{x^2+4x+8}\,dx.$$

The first integral is an easy substitution problem, using $u = x^2 + 4x + 8$:

$$\int \frac{x+2}{x^2+4x+8}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|x^2+4x+8|.$$

For the second integral we complete the square:

$$x^2+4x+8 = (x+2)^2+4 = 4\left(\left(\frac{x+2}{2}\right)^2+1\right),$$

making the integral

$$\frac{1}{4}\int \frac{1}{\left(\frac{x+2}{2}\right)^2+1}\,dx.$$

Using $u = \dfrac{x+2}{2}$ we get

$$\frac{1}{4}\int \frac{1}{\left(\frac{x+2}{2}\right)^2+1}\,dx = \frac{1}{4}\int \frac{2}{u^2+1}\,dx = \frac{1}{2}\arctan\left(\frac{x+2}{2}\right).$$

The final answer is now

$$\int \frac{x+1}{x^2+4x+8}\,dx = \frac{1}{2}\ln|x^2+4x+8| - \frac{1}{2}\arctan\left(\frac{x+2}{2}\right) + C.$$

♣

Exercises for 3.4

Exercise 3.4.1 $\displaystyle\int \frac{1}{4-x^2}\,dx$

Exercise 3.4.2 $\displaystyle\int \frac{x^4}{4-x^2}\,dx$

Exercise 3.4.3 $\displaystyle\int \frac{1}{x^2+10x+25}\,dx$

Exercise 3.4.4 $\displaystyle\int \frac{x^2}{4-x^2}\,dx$

Exercise 3.4.5 $\displaystyle\int \frac{x^4}{4+x^2}\,dx$

Exercise 3.4.6 $\displaystyle\int \frac{1}{x^2+10x+29}\,dx$

Exercise 3.4.7 $\displaystyle\int \frac{x^3}{4+x^2}\,dx$

Exercise 3.4.8 $\displaystyle\int \frac{1}{x^2+10x+21}\,dx$

Exercise 3.4.9 $\displaystyle\int \frac{1}{2x^2-x-3}\,dx$

Exercise 3.4.10 $\displaystyle\int \frac{1}{x^2+3x}\,dx$

Evaluate the indefinite integral.

Exercise 3.4.11 $\displaystyle\int \frac{7x+7}{x^2+3x-10}\,dx$

Exercise 3.4.12 $\displaystyle\int \frac{7x-2}{x^2+x}\,dx$

Exercise 3.4.13 $\displaystyle\int \frac{-4}{3x^2-12}\,dx$

Exercise 3.4.14 $\displaystyle\int \frac{x+7}{(x+5)^2}\,dx$

Exercise 3.4.15 $\displaystyle\int \frac{-3x-20}{(x+8)^2}\,dx$

Exercise 3.4.16 $\displaystyle\int \frac{9x^2+11x+7}{x(x+1)^2}\,dx$

Exercise 3.4.17 $\displaystyle\int \frac{-12x^2-x+33}{(x-1)(x+3)(3-2x)}\,dx$

Exercise 3.4.18 $\displaystyle\int \frac{94x^2-10x}{(7x+3)(5x-1)(3x-1)}\,dx$

Exercise 3.4.19 $\displaystyle\int \frac{x^2+x+1}{x^2+x-2}\,dx$

Exercise 3.4.20 $\displaystyle\int \frac{x^3}{x^2-x-20}\,dx$

Exercise 3.4.21 $\displaystyle\int \frac{2x^2-4x+6}{x^2-2x+3}\,dx$

Exercise 3.4.22 $\displaystyle\int \frac{1}{x^3+2x^2+3x}\,dx$

Exercise 3.4.23 $\displaystyle\int \frac{x^2+x+5}{x^2+4x+10}\,dx$

Exercise 3.4.24 $\displaystyle\int \frac{12x^2+21x+3}{(x+1)(3x^2+5x-1)}\,dx$

Exercise 3.4.25 $\displaystyle\int \frac{6x^2+8x-4}{(x-3)(x^2+6x+10)}\,dx$

Exercise 3.4.26 $\displaystyle\int \frac{2x^2+x+1}{(x+1)(x^2+9)}\,dx$

Exercise 3.4.27 $\displaystyle\int \frac{x^2-20x-69}{(x-7)(x^2+2x+17)}\,dx$

Exercise 3.4.28 $\displaystyle\int \frac{9x^2-60x+33}{(x-9)(x^2-2x+11)}\,dx$

Exercise 3.4.29 $\displaystyle\int \frac{6x^2+45x+121}{(x+2)(x^2+10x+27)}\,dx$

Evaluate the definite integral.

Exercise 3.4.30 $\displaystyle\int_1^2 \frac{8x+21}{(x+2)(x+3)}\,dx$

Exercise 3.4.31 $\displaystyle\int_0^5 \frac{14x+6}{(3x+2)(x+4)}\,dx$

Exercise 3.4.32 $\displaystyle\int_{-1}^1 \frac{x^2+5x-5}{(x-10)(x^2+4x+5)}\,dx$

Exercise 3.4.33 $\displaystyle\int_0^1 \frac{x}{(x+1)(x^2+2x+1)}\,dx$

3.5 Riemann Sums

A fundamental calculus technique is to first answer a given problem with an approximation, then refine that approximation to make it better, then use limits in the refining process to find the exact answer. That is exactly what we will do here to develop a technique to find the area of more complicated regions.

Consider the region given in Figure 3.6, which is the area under $y = 4x - x^2$ on $[0,4]$. What is the signed area of this region? While we will not use this notation in this section, we will soon see that this is equivalent to finding the integral given by $\int_0^4 (4x - x^2)\, dx$,

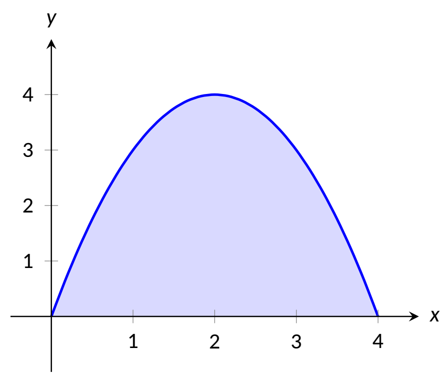


Figure 3.6: $f(x) = 4x - x^2$

We start by approximating. We can surround the region with a rectangle with height and width of 4 and find the area is approximately 16 square units. This is obviously an *over–approximation*; we are including area in the rectangle that is not under the parabola. How can we refine our approximation to make it better? The key to this section is this answer: *use more rectangles.*

Let's use four rectangles of equal width of 1. This partitions the interval $[0,4]$ into four *subintervals*, $[0,1]$, $[1,2]$, $[2,3]$ and $[3,4]$. On each subinterval we will draw a rectangle.

There are three common ways to determine the height of these rectangles: the **Left Hand Rule**, the **Right Hand Rule**, and the **Midpoint Rule**. The **Left Hand Rule** says to evaluate the function at the left–hand endpoint of the subinterval and make the rectangle that height. In Figure 3.7 below, the rectangle drawn on the interval $[2,3]$ has height determined by the Left Hand Rule; it has a height of $f(2) = 4$. (The rectangle is labeled "LHR.")

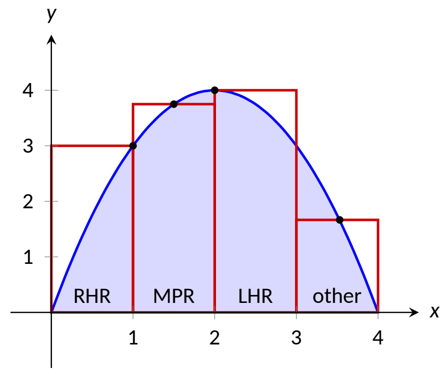


Figure 3.7: Approximating area using rectangles

The **Right Hand Rule** says the opposite: on each subinterval, evaluate the function at the right endpoint and make the rectangle that height. In the figure, the rectangle drawn on $[0,1]$ is drawn using $f(1) = 3$ as its height; this rectangle is labeled "RHR.".

The **Midpoint Rule** says that on each subinterval, evaluate the function at the midpoint and make the rectangle that height. The rectangle drawn on $[1,2]$ was made using the Midpoint Rule, with a height of $f(1.5) = 3.75$. That rectangle is labeled "MPR."

These are the three most common rules for determining the heights of approximating rectangles, but we are not forced to use one of these three methods. The rectangle on $[3,4]$ has a height of approximately $f(3.53)$, very close to the Midpoint Rule. It was chosen so that the area of the rectangle is *exactly* the area of the region under f on $[3,4]$. (Later you'll be able to figure how to do this, too.)

The following example will approximate the area under $f(x) = 4x - x^2$ using these rules.

Example 3.64: Using the Left Hand, Right Hand and Midpoint Rules

Approximate the area under $f(x) = 4x - x^2$ on the interval $[0,4]$ using the Left Hand Rule, the Right Hand Rule, and the Midpoint Rule, using four equally spaced subintervals.

Solution. We break the interval $[0,4]$ into four subintervals as before. In Figure 3.8 we see four rectangles drawn on $f(x) = 4x - x^2$ using the Left Hand Rule. (The areas of the rectangles are given in each figure.)

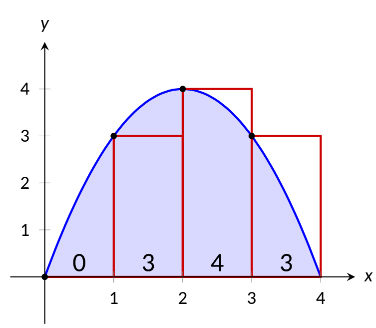


Figure 3.8: Approximating area using the Left Hand Rule

Note how in the first subinterval, $[0,1]$, the rectangle has height $f(0)=0$. We add up the areas of each rectangle (height× width) for our Left Hand Rule approximation:

$$f(0)\cdot 1+f(1)\cdot 1+f(2)\cdot 1+f(3)\cdot 1= \\ 0+3+4+3=10.$$

Figure 3.9 shows four rectangles drawn under $f(x)$ using the Right Hand Rule; note how the $[3,4]$ subinterval has a rectangle of height 0.

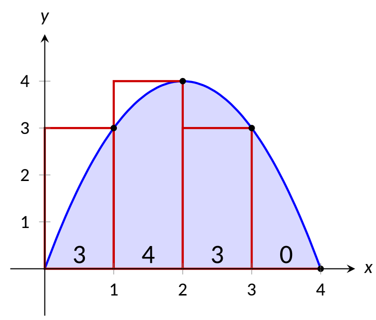


Figure 3.9: Approximating area using the Right Hand Rule,

In this figure, these rectangles seem to be the mirror image of those found in Figure 3.8. (This is because of the symmetry of our shaded region.) Our approximation gives the same answer as before, though calculated a different way:

$$f(1)\cdot 1+f(2)\cdot 1+f(3)\cdot 1+f(4)\cdot 1= \\ 3+4+3+0=10.$$

Figure 3.10 shows four rectangles drawn under $f(x)$ using the Midpoint Rule.

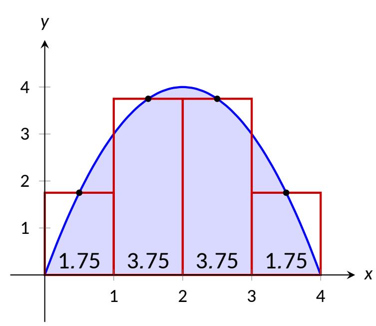


Figure 3.10: Approximating area using the Midpoint Rule

This gives an approximation of the area as:

$$f(0.5)\cdot 1+f(1.5)\cdot 1+f(2.5)\cdot 1+f(3.5)\cdot 1= \\ 1.75+3.75+3.75+1.75=11.$$

Our three methods provide two approximations of the area under $f(x) = 4x - x^2$: 10 and 11. ♣

It is hard to tell at this moment which is a better approximation: 10 or 11? We can continue to refine our approximation by using more rectangles. The notation can become unwieldy, though, as we add up longer and longer lists of numbers. We introduce **summation notation** (also called **sigma notation**) to solve this problem.

Suppose we wish to add up a list of numbers $a_1, a_2, a_3, \ldots, a_9$. Instead of writing

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9,$$

we use summation notation and write

$$\sum_{i=1}^{9} a_i$$

The upper case sigma ($\sum$) represents the term "sum" and a_i is referred to as the summand. The index of summation in this example is i; any symbol can be used. By convention, the index takes on only the integer values between (and including) the lower and upper bounds, here equal to 1 and 9 respectively.

Let's practice using this notation.

Example 3.65: Using Summation Notation

Let the numbers $\{a_i\}$ be defined as $a_i = 2i - 1$ for integers i, where $i \geq 1$. So $a_1 = 1$, $a_2 = 3$, $a_3 = 5$, etc. (The output is the positive odd integers). Evaluate the following summations:

1. $\sum_{i=1}^{6} a_i$
2. $\sum_{i=3}^{7} (3a_i - 4)$
3. $\sum_{i=1}^{4} (a_i)^2$

Solution.

1.

$$\begin{aligned}\sum_{i=1}^{6} a_i &= a_1 + a_2 + a_3 + a_4 + a_5 + a_6 \\ &= 1 + 3 + 5 + 7 + 9 + 11 \\ &= 36.\end{aligned}$$

2. Note the starting value is different than 1:

$$\begin{aligned}\sum_{i=3}^{7} (3a_i - 4) &= (3a_3 - 4) + (3a_4 - 4) + (3a_5 - 4) + (3a_6 - 4) + (3a_7 - 4) \\ &= 11 + 17 + 23 + 29 + 35 \\ &= 115.\end{aligned}$$

3.

$$\begin{aligned}\sum_{i=1}^{4} (a_i)^2 &= (a_1)^2 + (a_2)^2 + (a_3)^2 + (a_4)^2 \\ &= 1^2 + 3^2 + 5^2 + 7^2 \\ &= 84\end{aligned}$$

♣

The following theorem gives some properties of summations that allow us to work with them without writing individual terms. Examples will follow.

Theorem 3.66: Properties of Summations

1. $\displaystyle\sum_{i=1}^{n} c = c \cdot n$, *where c is a constant.*
2. $\displaystyle\sum_{i=m}^{n} (a_i \pm b_i) = \sum_{i=m}^{n} a_i \pm \sum_{i=m}^{n} b_i$
3. $\displaystyle\sum_{i=m}^{n} c \cdot a_i = c \cdot \sum_{i=m}^{n} a_i$
4. $\displaystyle\sum_{i=m}^{j} a_i + \sum_{i=j+1}^{n} a_i = \sum_{i=m}^{n} a_i$
5. $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
6. $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
7. $\displaystyle\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

Example 3.67: Evaluating Summations

Use Theorem 3.66 to evaluate

$$\sum_{i=1}^{6} a_i = \sum_{i=1}^{6} (2i-1)$$

where i are integers and $i \geq 1$.

Solution.

$$\begin{aligned}\sum_{i=1}^{6} (2i-1) &= \sum_{i=1}^{6} 2i - \sum_{i=1}^{6} (1) \\ &= \left(2\sum_{i=1}^{6} i\right) - 6 \\ &= 2\frac{6(6+1)}{2} - 6 \\ &= 42 - 6 = 36\end{aligned}$$

We obtained the same answer without writing out all six terms. When dealing with small values of n, it may be faster to write the terms out by hand. However, Theorem 3.66 is incredibly important when dealing with large sums as we'll soon see. ♣

Consider again $f(x) = 4x - x^2$. We will approximate the area under this curve (again for $[0,4]$ using sixteen equally spaced subintervals and the Right Hand Rule. Before doing so, we will do some careful preparation.

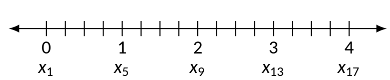


Figure 3.11: Dividing $[0,4]$ into 16 equally spaced subintervals

Figure 3.11 shows a number line of $[0,4]$ divided into sixteen equally spaced subintervals. We denote 0 as x_1; we have marked the values of x_5, x_9, x_{13} and x_{17}. We could mark them all, but the figure would get crowded. While it is easy to figure that $x_{10} = 2.25$, in general, we want a method of determining the value of x_i without consulting the figure. Consider:

$$\begin{aligned} x_i &= x_1 + (i-1)\Delta x \\ \textit{where} & \\ x_1 &: \text{ starting value} \\ (i-1) &: \text{ number of subintervals between } x_1 \text{ and } x_i \\ \Delta x &: \text{ subinterval width} \end{aligned}$$

So $x_{10} = x_1 + 9(4/16) = 2.25$.

If we had partitioned $[0,4]$ into 100 equally spaced subintervals, each subinterval would have length $\Delta x = 4/100 = 0.04$. We could compute x_{32} as

$$x_{32} = 0 + 31(4/100) = 1.24.$$

(That was far faster than creating a sketch first.)

Given *any* subdivision of $[0,4]$, the first subinterval is $[x_1, x_2]$; the second is $[x_2, x_3]$; the i^{th} subinterval is $[x_i, x_{i+1}]$.

When using the Left Hand Rule, the height of the i^{th} rectangle will be $f(x_i)$.

When using the Right Hand Rule, the height of the i^{th} rectangle will be $f(x_{i+1})$.

When using the Midpoint Rule, the height of the i^{th} rectangle will be $f\left(\dfrac{x_i + x_{i+1}}{2}\right)$.

Thus approximating the area under $f(x) = 4x - x^2$ on $[0,4]$ with sixteen equally spaced subintervals can be expressed as follows, where $\Delta x = 4/16 = 1/4$:

Left Hand Rule: $\displaystyle\sum_{i=1}^{16} f(x_i)\Delta x$

Right Hand Rule: $\displaystyle\sum_{i=1}^{16} f(x_{i+1})\Delta x$

Midpoint Rule: $\displaystyle\sum_{i=1}^{16} f\left(\frac{x_i + x_{i+1}}{2}\right)\Delta x$

We use these formulas in the following example.

Example 3.68: Approximating Area Using Sums

Approximate the area under $f(x) = 4x - x^2$ on $[0,4]$ using the Right Hand Rule and summation formulas with sixteen and 1000 equally spaced intervals.

Solution. Using sixteen equally spaced intervals and the Right Hand Rule, we can approximate the area as

$$\sum_{i=1}^{16} f(x_{i+1})\Delta x.$$

We have $\Delta x = 4/16 = 0.25$. Since $x_i = 0 + (i-1)\Delta x$, we have

$$x_{i+1} = 0 + \big((i+1) - 1\big)\Delta x$$

$$= i\Delta x$$

Using the summation formulas, consider:

$$\begin{aligned}
\sum_{i=1}^{16} f(x_{i+1})\Delta x &= \sum_{i=1}^{16} f(i\Delta x)\Delta x \\
&= \sum_{i=1}^{16} \left(4i\Delta x - (i\Delta x)^2\right)\Delta x \\
&= \sum_{i=1}^{16} (4i\Delta x^2 - i^2\Delta x^3) \\
&= (4\Delta x^2)\sum_{i=1}^{16} i - \Delta x^3 \sum_{i=1}^{16} i^2 \qquad (3.2) \\
&= (4\Delta x^2)\frac{16\cdot 17}{2} - \Delta x^3 \frac{16(17)(33)}{6} \\
&= 4\cdot 0.25^2\cdot 136 - 0.25^3\cdot 1496 \\
&= 10.625
\end{aligned}$$

We were able to sum up the areas of sixteen rectangles with very little computation. In Figure 3.12 the function and the sixteen rectangles are graphed. While some rectangles over–approximate the area, other under–approximate the area (by about the same amount). Thus our approximate area of 10.625 is likely a fairly good approximation.

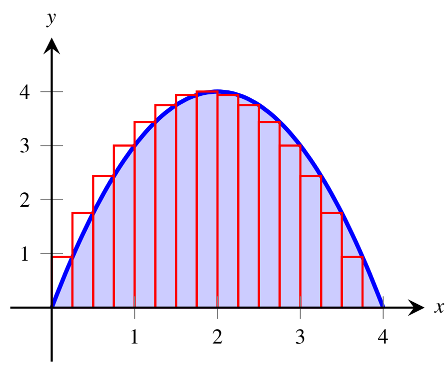


Figure 3.12: Approximating area with the Right Hand Rule and 16 evenly spaced subintervals

Notice Equation (3.2); by changing the 16's to 1,000's (and appropriately changing the value of Δx), we can use that equation to sum up 1000 rectangles!

We do so here, skipping from the original summand to the equivalent of Equation (3.2) to save space. Note that $\Delta x = 4/1000 = 0.004$.

$$\begin{aligned}
\sum_{i=1}^{1000} f(x_{i+1})\Delta x &= (4\Delta x^2)\sum_{i=1}^{1000} i - \Delta x^3 \sum_{i=1}^{1000} i^2 \\
&= (4\Delta x^2)\frac{1000\cdot 1001}{2} - \Delta x^3 \frac{1000(1001)(2001)}{6} \\
&= 4\cdot 0.004^2\cdot 500500 - 0.004^3\cdot 333,833,500 \\
&= 10.666656
\end{aligned}$$

Using many, many rectangles, we have a likely good approximation of the area under $f(x) = 4x - x^2$ of ≈ 10.666656. ♣

Before the above example, we stated the summations for the Left Hand, Right Hand and Midpoint Rules. Each had the same basic structure, which was:

1. each rectangle has the same width, which we referred to as Δx, and
2. each rectangle's height is determined by evaluating $f(x)$ at a particular point in each subinterval. For instance, the Left Hand Rule states that each rectangle's height is determined by evaluating $f(x)$ at the left hand endpoint of the subinterval the rectangle lives on.

One could partition an interval $[a,b]$ with subintervals that did not have the same width. We refer to the length of the first subinterval as Δx_1, the length of the second subinterval as Δx_2, and so on, giving the length of the i^{th} subinterval as Δx_i. Also, one could determine each rectangle's height by evaluating $f(x)$ at *any* point in the i^{th} subinterval. We refer to the point picked in the first subinterval as c_1, the point picked in the second subinterval as c_2, and so on, with c_i representing the point picked in the i^{th} subinterval. Thus the height of the i^{th} subinterval would be $f(c_i)$, and the area of the i^{th} rectangle would be $f(c_i)\Delta x_i$.

Summations of rectangles with area $f(c_i)\Delta x_i$ are named after mathematician Georg Friedrich Bernhard Riemann, as given in the following definition.

Definition 3.69: Riemann Sum

Let $f(x)$ be defined on the closed interval $[a,b]$ and let Δx be a partition of $[a,b]$, with

$$a = x_1 < x_2 < \ldots < x_n < x_{n+1} = b.$$

Let Δx_i denote the length of the i^{th} subinterval $[x_i, x_{i+1}]$ and let c_i denote any value in the i^{th} subinterval. The sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i$$

*is a **Riemann sum** of $f(x)$ on $[a,b]$.*

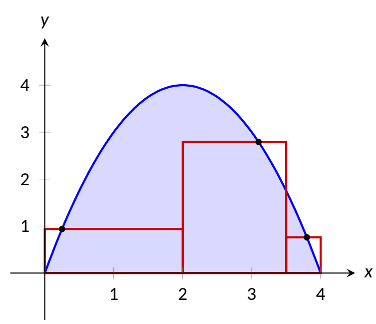


Figure 3.13: General Riemann sum to approximate the area under $f(x) = 4x - x^2$

Figure 3.13 shows the approximating rectangles of a Riemann sum. While the rectangles in this example do not approximate well the shaded area, they demonstrate that the subinterval widths may vary and the heights of the rectangles can be determined without following a particular rule.

Riemann sums are typically calculated using one of the three rules we have introduced. The uniformity of construction makes computations easier. Before working another example, let's summarize some of what we have learned in a convenient way.

Riemann Sums

Consider a function $f(x)$ defined on an interval $[a,b]$. The area under this curve is approximated by $\sum_{i=1}^{n} f(c_i)\Delta x_i$.

1. When the n subintervals have equal length, $\Delta x_i = \Delta x = \dfrac{b-a}{n}$.
2. The i^{th} term of the partition is $x_i = a + (i-1)\Delta x$. (This makes $x_{n+1} = b$.)
3. The Left Hand Rule summation is: $\displaystyle\sum_{i=1}^{n} f(x_i)\Delta x$.
4. The Right Hand Rule summation is: $\displaystyle\sum_{i=1}^{n} f(x_{i+1})\Delta x$.
5. The Midpoint Rule summation is: $\displaystyle\sum_{i=1}^{n} f\left(\frac{x_i + x_{x+1}}{2}\right)\Delta x$.

Let's do another example.

Example 3.70: Approximating Area Using Sums

Approximate the area under $f(x) = (5x+2)$ on the interval $[-2,3]$ using the Midpoint Rule and ten equally spaced intervals.

Solution. Following the above discussion, we have

$$\Delta x = \frac{3-(-2)}{10} = 1/2$$
$$x_i = (-2) + (1/2)(i-1) = i/2 - 5/2.$$

As we are using the Midpoint Rule, we will also need x_{i+1} and $\dfrac{x_i + x_{i+1}}{2}$. Since $x_i = i/2 - 5/2$, $x_{i+1} = (i+1)/2 - 5/2 = i/2 - 2$. This gives

$$\frac{x_i + x_{i+1}}{2} = \frac{(i/2 - 5/2) + (i/2 - 2)}{2} = \frac{i - 9/2}{2} = i/2 - 9/4.$$

We now construct the Riemann sum and compute its value using summation formulas.

$$\begin{aligned}
\sum_{i=1}^{10} f\left(\frac{x_i + x_{i+1}}{2}\right)\Delta x &= \sum_{i=1}^{10} f(i/2 - 9/4)\Delta x \\
&= \sum_{i=1}^{10} (5(i/2 - 9/4) + 2)\Delta x \\
&= \Delta x \sum_{i=1}^{10} \left[\left(\frac{5}{2}\right) i - \frac{37}{4}\right] \\
&= \Delta x \left(\frac{5}{2}\sum_{i=1}^{10}(i) - \sum_{i=1}^{10}\left(\frac{37}{4}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left(\frac{5}{2}\cdot\frac{10(11)}{2} - 10\cdot\frac{37}{4}\right)\\
&= \frac{45}{2} = 22.5
\end{aligned}
$$

Note the graph of $f(x) = 5x+2$ in Figure 3.14. The regions whose areas are computed are triangles, meaning we can find the exact answer without summation techniques. We find that the exact answer is indeed 22.5. One of the strengths of the Midpoint Rule is that often each rectangle includes area that should not be counted, but misses other area that should. When the partition width is small, these two amounts are about equal and these errors almost "cancel each other out." In this example, since our function is a line, these errors are exactly equal and they do cancel each other out, giving us the exact answer.

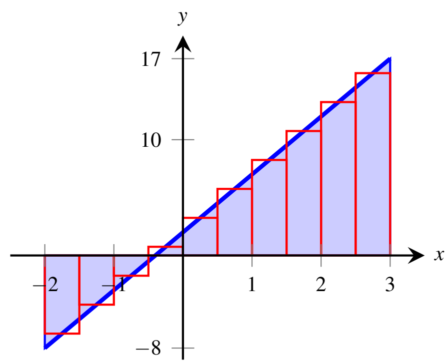


Figure 3.14: Approximating area using the Midpoint Rule and 10 evenly spaced subintervals

Note too that when the function is negative, the rectangles have a "negative" height. When we compute the area of the rectangle, we use $f(c_i)\Delta x$; when f is negative, the area is counted as negative. ♣

Notice in the previous example that while we used ten equally spaced intervals, the number "10" didn't play a big role in the calculations until the very end. Mathematicians love abstract ideas; let's approximate the area of another region using n subintervals, where we do not specify a value of n until the very end.

Example 3.71: Approximating Area Using Sums

Revisit $f(x) = 4x - x^2$ on the interval $[0,4,]$ yet again. Approximate the area under this curve using the Right Hand Rule with n equally spaced subintervals.

Solution. We know $\Delta x = \frac{4-0}{n} = 4/n$. We also find $x_i = 0 + \Delta x(i-1) = 4(i-1)/n$. The Right Hand Rule uses x_{i+1}, which is $x_{i+1} = 4i/n$. We construct the Right Hand Rule Riemann sum as follows.

$$
\begin{aligned}
\sum_{i=1}^{n} f(x_{i+1})\Delta x &= \sum_{i=1}^{n} f\left(\frac{4i}{n}\right)\Delta x\\
&= \sum_{i=1}^{n}\left[4\frac{4i}{n} - \left(\frac{4i}{n}\right)^2\right]\Delta x\\
&= \sum_{i=1}^{n}\left(\frac{16\Delta x}{n}\right)i - \sum_{i=1}^{n}\left(\frac{16\Delta x}{n^2}\right)i^2\\
&= \left(\frac{16\Delta x}{n}\right)\sum_{i=1}^{n} i - \left(\frac{16\Delta x}{n^2}\right)\sum_{i=1}^{n} i^2\\
&= \left(\frac{16\Delta x}{n}\right)\cdot\frac{n(n+1)}{2} - \left(\frac{16\Delta x}{n^2}\right)\frac{n(n+1)(2n+1)}{6} \qquad \left(\begin{array}{c}\text{recall}\\ \Delta x = 4/n\end{array}\right)
\end{aligned}
$$

$$= \frac{32(n+1)}{n} - \frac{32(n+1)(2n+1)}{3n^2} \quad \text{(now simplify)}$$
$$= \frac{32}{3}\left(1 - \frac{1}{n^2}\right)$$

The result is an amazing, easy to use formula. To approximate the area with ten equally spaced subintervals and the Right Hand Rule, set $n = 10$ and compute

$$\frac{32}{3}\left(1 - \frac{1}{10^2}\right) = 10.56.$$

Recall how earlier we approximated the area with 4 subintervals; with $n = 4$, the formula gives 10, our answer as before.

It is now easy to approximate the area with 1,000,000 subintervals! Hand-held calculators will round off the answer a bit prematurely giving an answer of 10.66666667. (The actual answer is 10.666666666656.)

We now take an important leap. Up to this point, our mathematics has been limited to geometry and algebra (finding areas and manipulating expressions). Now we apply *calculus*. For any *finite* n, we know that the corresponding Right Hand Rule Riemann sum is:

$$\frac{32}{3}\left(1 - \frac{1}{n^2}\right).$$

Both common sense and high–level mathematics tell us that as n gets large, the approximation gets better. In fact, if we take the *limit* as $n \to \infty$, we get the *exact area*. That is,

$$\lim_{n\to\infty} \frac{32}{3}\left(1 - \frac{1}{n^2}\right) = \frac{32}{3}(1-0)$$
$$= \frac{32}{3} = 10.\overline{6}$$

This is a fantastic result. By considering n equally–spaced subintervals, we obtained a formula for an approximation of the area that involved our variable n. As n grows large – without bound – the error shrinks to zero and we obtain the exact area. ♣

This section started with a fundamental calculus technique: make an approximation, refine the approximation to make it better, then use limits in the refining process to get an exact answer. That is precisely what we just did.

Let's practice this again.

Example 3.72: Approximating Area With a Formula, Using Sums

Find a formula that approximates the area under $f(x) = x^3$ on the interval $[-1,5]$ using the Right Hand Rule and n equally spaced subintervals, then take the limit as $n \to \infty$ to find the exact area.

Solution. We have $\Delta x = \frac{5-(-1)}{n} = 6/n$. We have $x_i = (-1) + (i-1)\Delta x$; as the Right Hand Rule uses x_{i+1}, we have $x_{i+1} = (-1) + i\Delta x$.

The Riemann sum corresponding to the Right Hand Rule is (followed by simplifications):

$$\sum_{i=1}^{n} f(x_{i+1})\Delta x = \sum_{i=1}^{n} f(-1 + i\Delta x)\Delta x$$
$$= \sum_{i=1}^{n} (-1 + i\Delta x)^3 \Delta x$$

$$\begin{aligned}
&= \sum_{i=1}^{n} \big((i\Delta x)^3 - 3(i\Delta x)^2 + 3i\Delta x - 1\big)\Delta x \\
&= \sum_{i=1}^{n} \big(i^3\Delta x^4 - 3i^2\Delta x^3 + 3i\Delta x^2 - \Delta x\big) \\
&= \Delta x^4 \sum_{i=1}^{n} i^3 - 3\Delta x^3 \sum_{i=1}^{n} i^2 + 3\Delta x^2 \sum_{i=1}^{n} i - \sum_{i=1}^{n} \Delta x \\
&= \Delta x^4 \left(\frac{n(n+1)}{2}\right)^2 - 3\Delta x^3 \frac{n(n+1)(2n+1)}{6} + 3\Delta x^2 \frac{n(n+1)}{2} - n\Delta x \\
&= \frac{1296}{n^4} \cdot \frac{n^2(n+1)^2}{4} - 3\frac{216}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + 3\frac{36}{n^2}\frac{n(n+1)}{2} - 6 \\
&= 156 + \frac{378}{n} + \frac{216}{n^2}
\end{aligned}$$

Once again, we have found a compact formula for approximating the area with n equally spaced subintervals and the Right Hand Rule. Using ten subintervals, we have an approximation of 195.96 (these rectangles are shown in Figure 3.15). Using $n = 100$ gives an approximation of 159.802.

Now find the exact answer using a limit:

$$\lim_{n\to\infty} \left(156 + \frac{378}{n} + \frac{216}{n^2}\right) = 156.$$

♣

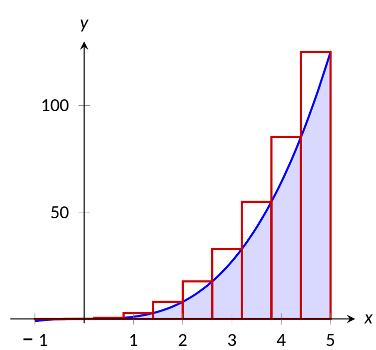


Figure 3.15: Approximating area using the Right Hand Rule and 10 evenly spaced subintervals.

We have used limits to evaluate exactly given definite limits. Will this always work? We will show, given not–very–restrictive conditions, that yes, it will always work.

The previous two examples demonstrated how an expression such as

$$\sum_{i=1}^{n} f(x_{i+1})\Delta x$$

can be rewritten as an expression explicitly involving n, such as $32/3(1-1/n^2)$.

Viewed in this manner, we can think of the summation as a function of n. An n value is given (where n is a positive integer), and the sum of areas of n equally spaced rectangles is returned, using the Left Hand, Right Hand, or Midpoint Rules.

Given a function $f(x)$ defined on the interval $[a,b]$ let:

- $S_L(n) = \sum_{i=1}^{n} f(x_i)\Delta x$, the sum of equally spaced rectangles formed using the Left Hand Rule,
- $S_R(n) = \sum_{i=1}^{n} f(x_{i+1})\Delta x$, the sum of equally spaced rectangles formed using the Right Hand Rule, and
- $S_M(n) = \sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1}}{2}\right)\Delta x$, the sum of equally spaced rectangles formed using the Midpoint Rule.

Recall the definition of a limit as $n \to \infty$: $\lim_{n\to\infty} S_L(n) = K$ if, given any $\varepsilon > 0$, there exists $\delta > 0$ such that $|S_L(n) - K| < \varepsilon$ whenever $n \geq \delta$.

The following theorem states that we can use any of our three rules to find the exact value of the area under $f(x)$ on $[a,b]$. It also goes two steps further. The theorem states that the height of each rectangle doesn't have to be determined following a specific rule, but could be $f(c_i)$, where c_i is any point in the i^{th} subinterval, as discussed earlier.

The theorem goes on to state that the rectangles do not need to be of the same width. Using the notation of Definition 3.69, let Δx_i denote the length of the i^{th} subinterval in a partition of $[a,b]$. Now let $||\Delta x||$ represent the length of the largest subinterval in the partition: that is, $||\Delta x||$ is the largest of all the Δx_i's. If $||\Delta x||$ is small, then $[a,b]$ must be partitioned into many subintervals, since all subintervals must have small lengths. "Taking the limit as $||\Delta x||$ goes to zero" implies that the number n of subintervals in the partition is growing to infinity, as the largest subinterval length is becoming arbitrarily small. We then interpret the expression

$$\lim_{||\Delta x||\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

as "the limit of the sum of rectangles, where the width of each rectangle can be different but getting small, and the height of each rectangle is not necessarily determined by a particular rule." The following theorem states that, for a sufficiently nice function, we can use any of our three rules to find the area under $f(x)$ over $[a,b]$.

Theorem 3.73: Area and the Limit of Riemann Sums

Let $f(x)$ be a continuous function on the closed interval $[a,b]$ and let $S_L(n)$, $S_R(n)$ and $S_M(n)$ be the sums of equally spaced rectangles formed using the Left Hand Rule, Right Hand Rule, and Midpoint Rule, respectively. Then:

1. $\lim_{n\to\infty} S_L(n) = \lim_{n\to\infty} S_R(n) = \lim_{n\to\infty} S_M(n) = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$
2. *The area under f on the interval $[a,b]$ is equal to* $\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$.
3. *The area under f on the interval $[a,b]$ is equal to* $\lim_{||\Delta x||\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$.

We summarize what we have learned over the past few sections here.

- Knowing the "area under the curve" can be useful. One common example is: the area under a velocity curve is displacement.
- While we can approximate the area under a curve in many ways, we have focused on using rectangles whose heights can be determined using: the Left Hand Rule, the Right Hand Rule and the Midpoint Rule.

- Sums of rectangles of this type are called Riemann sums.
- The exact value of the area can be computed using the limit of a Riemann sum. We generally use one of the above methods as it makes the algebra simpler.

Exercises for Section 3.5

In the following, write out each term of the summation and compute the sum.

Exercise 3.5.1 $\sum_{i=2}^{4} i^2$

Exercise 3.5.2 $\sum_{i=-1}^{3} (4i-2)$

Exercise 3.5.3 $\sum_{i=-2}^{2} \sin(\pi i/2)$

Exercise 3.5.4 $\sum_{i=1}^{5} \frac{1}{i}$

Exercise 3.5.5 $\sum_{i=1}^{6} (-1)^i i$

Exercise 3.5.6 $\sum_{i=1}^{4} \left(\frac{1}{i} - \frac{1}{i+1}\right)$

Exercise 3.5.7 $\sum_{i=0}^{5} (-1)^i \cos(\pi i)$

Write each sum in summation notation.

Exercise 3.5.8 $3+6+9+12+15$

Exercise 3.5.9 $-1+0+3+8+15+24+35+48+63$

Exercise 3.5.10 $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}$

Exercise 3.5.11 $1-e+e^2-e^3+e^4$

Evalaute the summation.

Exercise 3.5.12 $\sum_{i=1}^{25} i$

Exercise 3.5.13 $\sum_{i=1}^{10} (3i^2-2i)$

Exercise 3.5.14 $\sum_{i=1}^{15} (2i^3-10)$

Exercise 3.5.15 $\sum_{i=1}^{10} (-4i^3+10i^2-7i+11)$

Exercise 3.5.16 $\sum_{i=1}^{10} (i^3-3i^2+2i+7)$

Exercise 3.5.17 $1+2+3+\ldots+99+100$

Exercise 3.5.18 $1+4+9+\ldots+361+400$

In each of the following exercises, a definite integral $\int_a^b f(x)\,dx$ is given.

(a) Graph $f(x)$ on $[a,b]$.

(b) Add to the sketch rectangles using the provided rule.

(c) Approximate $\int_a^b f(x)\,dx$ by summing the areas of the rectangles.

Exercise 3.5.19 $\int_{-3}^{3} x^2\,dx$, *with 6 rectangles using the Left Hand Rule.*

Exercise 3.5.20 $\int_{0}^{2} (5-x^2)\,dx$, *with 4 rectangles using the Midpoint Rule.*

Exercise 3.5.21 $\int_{0}^{\pi} \sin x\,dx$, *with 6 rectangles using the Right Hand Rule.*

Exercise 3.5.22 $\int_{0}^{3} 2^x\,dx$, *with 5 rectangles using the Left Hand Rule*

Exercise 3.5.23 $\int_{1}^{2} \ln x\,dx$, *with 3 rectangles using the Midpoint Rule.*

Exercise 3.5.24 $\int_{1}^{9} \frac{1}{x}\,dx$, *with 4 rectangles using the Right Hand Rule.*

In the following exercises, a definite integral $\int_a^b f(x)\,dx$ is given. As demonstrated in the section, do the following.

(a) Find a formula to approximate $\int_a^b f(x)\,dx$ using n subintervals and the provided rule.

(b) Evaluate the formula using $n = 10$, 100 and 1,000.

(c) Find the limit of the formula, as $n \to \infty$, to find the exact value of $\int_a^b f(x)\,dx$.

Exercise 3.5.25 $\int_{0}^{1} x^3\,dx$, *using the Right Hand Rule.*

Exercise 3.5.26 $\int_{-1}^{1} 3x^2\,dx$, *using the Left Hand Rule.*

Exercise 3.5.27 $\int_{-1}^{3} (3x-1)\,dx$, *using the Midpoint Rule.*

Exercise 3.5.28 $\int_{1}^{4} (2x^2-3)\,dx$, *using the Left Hand Rule.*

Exercise 3.5.29 $\int_{-10}^{10} (5-x)\,dx$, *using the Right Hand Rule.*

Exercise 3.5.30 $\int_{0}^{1} (x^3-x^2)\,dx$, *using the Right Hand Rule.*

3.6 Numerical Integration

We have now seen some of the most generally useful methods for discovering antiderivatives, and there are others. Unfortunately, some functions have no simple antiderivatives. In such cases, if the value of a definite integral is needed it will have to be approximated. We will see two methods that work reasonably well and yet are fairly simple; in some cases more sophisticated techniques will be needed.

Start with "one way to approximate an integral is with rectangles". Of course, we already know one way to approximate an integral: If we think of the integral as computing an area, we can add up the areas of some rectangles.

Midpoint rule.

While this method is quite simple, it is usually the case that a large number of rectangles is needed to get acceptable accuracy. A similar approach is much better. We approximate the area under a curve over a small interval as the area of a trapezoid. In figure 3.16 we see an area under a curve approximated by rectangles and by trapezoids; it is apparent that the trapezoids give a substantially better approximation on each subinterval.

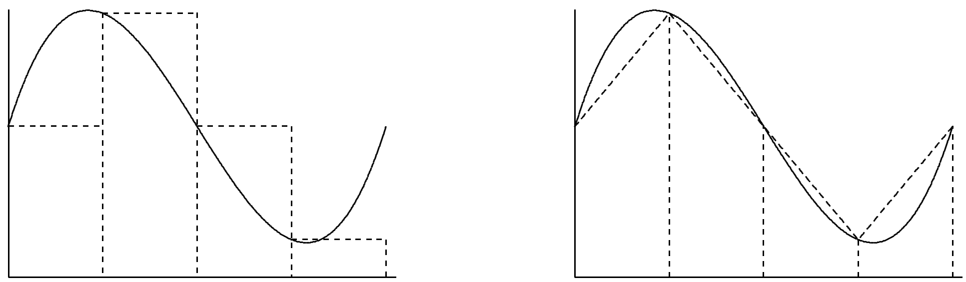


Figure 3.16: Approximating an area with rectangles and with trapezoids.

As with rectangles, we divide the interval into n equal subintervals of length Δx. A typical trapezoid is pictured in figure 3.17; it has area

$$\frac{f(x_i)+f(x_{i+1})}{2}\Delta x.$$

If we add up the areas of all trapezoids we get

$$\begin{aligned}\frac{f(x_0)+f(x_1)}{2}\Delta x + \frac{f(x_1)+f(x_2)}{2}\Delta x + \cdots + \frac{f(x_{n-1})+f(x_n)}{2}\Delta x \\ = \left(\frac{f(x_0)}{2}+f(x_1)+f(x_2)+\cdots+f(x_{n-1})+\frac{f(x_n)}{2}\right)\Delta x.\end{aligned}$$

For a modest number of subintervals this is not too difficult to do with a calculator; a computer can easily do many subintervals.

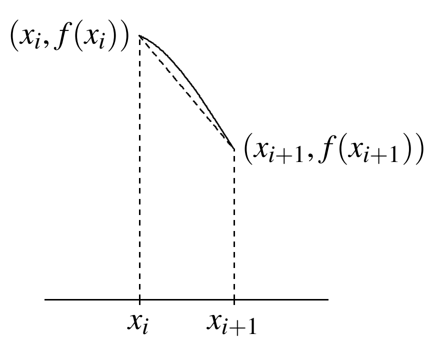


Figure 3.17: A single trapezoid.

In practice, an approximation is useful only if we know how accurate it is; for example, we might need a particular value accurate to three decimal places. When we compute a particular approximation to an integral, the error is the difference between the approximation and the true value of the integral. For any approximation technique, we need an **error bound**, a value that is guaranteed to be larger than the actual error. If A is an approximation and E is the associated error bound, then we know that the true value of the integral is between $A-E$ and $A+E$. In the case of our approximation of the integral, we want $E = E(\Delta x)$ to be a function of Δx that gets small rapidly as Δx gets small. Fortunately, for many functions, there is such an error bound associated with the trapezoid approximation.

Theorem 3.74: Error for Trapezoid Approximation

Suppose f has a second derivative f'' everywhere on the interval $[a,b]$, and $|f''(x)| \le M$ for all x in the interval. With $\Delta x = (b-a)/n$, an error bound for the trapezoid approximation is

$$E(\Delta x) = \frac{b-a}{12}M(\Delta x)^2 = \frac{(b-a)^3}{12n^2}M.$$

Let's see how we can use this.

Example 3.75: Approximate an Integral With Trapezoids

Approximate $\int_0^1 e^{-x^2}\,dx$ to two decimal places.

Solution. The second derivative of $f = e^{-x^2}$ is $(4x^2-2)e^{-x^2}$, and it is not hard to see that on $[0,1]$ $|f''(x)|$ has a maximum value of 2, thus we begin by estimating the number of subintervals we are likely to need. To get two decimal places of accuracy, we will certainly need $E(\Delta x) < 0.005$ or

$$\begin{aligned} \frac{1}{12}(2)\frac{1}{n^2} &< 0.005 \\ \frac{1}{6}(200) &< n^2 \\ 5.77 \approx \sqrt{\frac{100}{3}} &< n \end{aligned}$$

With $n=6$, the error bound is thus $1/6^3 < 0.0047$. We compute the trapezoid approximation for six intervals:

$$\left(\frac{f(0)}{2} + f(1/6) + f(2/6) + \cdots + f(5/6) + \frac{f(1)}{2}\right)\frac{1}{6} \approx 0.74512.$$

So the true value of the integral is between $0.74512 - 0.0047 = 0.74042$ and $0.74512 + 0.0047 = 0.74982$. Unfortunately, the first rounds to 0.74 and the second rounds to 0.75, so we can't be sure of the correct value in the second decimal place; we need to pick a larger n. As it turns out, we need to go to $n = 12$ to get two bounds that both round to the same value, which turns out to be 0.75. For comparison, using 12 rectangles to approximate the area gives 0.7727, which is considerably less accurate than the approximation using six trapezoids.

In practice it generally pays to start by requiring better than the maximum possible error; for example, we might have initially required $E(\Delta x) < 0.001$, or

$$\frac{1}{12}(2)\frac{1}{n^2} < 0.001$$

$$\begin{aligned}\frac{1}{6}(1000) &< n^2\\ 12.91 \approx \sqrt{\frac{500}{3}} &< n\end{aligned}$$

Had we immediately tried $n = 13$ this would have given us the desired answer. ♣

Compare trapezoid and midpoint rules

The trapezoid approximation works well, especially compared to rectangles, because the tops of the trapezoids form a reasonably good approximation to the curve when Δx is fairly small. We can extend this idea: what if we try to approximate the curve more closely by using something other than a straight line? The obvious candidate is a parabola: If we can approximate a short piece of the curve with a parabola with equation $y = ax^2 + bx + c$, we can easily compute the area under the parabola.

There are an infinite number of parabolas through any two given points, but only one through three given points. If we find a parabola through three consecutive points $(x_i, f(x_i))$, $(x_{i+1}, f(x_{i+1}))$, $(x_{i+2}, f(x_{i+2}))$ on the curve, it should be quite close to the curve over the whole interval $[x_i, x_{i+2}]$, as in Figure 3.18. If we divide the interval $[a,b]$ into an even number of subintervals, we can then approximate the curve by a sequence of parabolas, each covering two of the subintervals. For this to be practical, we would like a simple formula for the area under one parabola, namely, the parabola through $(x_i, f(x_i))$, $(x_{i+1}, f(x_{i+1}))$, and $(x_{i+2}, f(x_{i+2}))$. That is, we should attempt to write down the parabola $y = ax^2 + bx + c$ through these points and then integrate it, and hope that the result is fairly simple. Although the algebra involved is messy, this turns out to be possible. The algebra is well within the capability of a good computer algebra system like Sage, so we will present the result without all of the algebra.

To find the parabola, we solve these three equations for a, b, and c:

$$\begin{aligned}f(x_i) &= a(x_{i+1} - \Delta x)^2 + b(x_{i+1} - \Delta x) + c\\ f(x_{i+1}) &= a(x_{i+1})^2 + b(x_{i+1}) + c\\ f(x_{i+2}) &= a(x_{i+1} + \Delta x)^2 + b(x_{i+1} + \Delta x) + c\end{aligned}$$

Not surprisingly, the solutions turn out to be quite messy. Nevertheless, Sage can easily compute and simplify the integral to get

$$\int_{x_{i+1}-\Delta x}^{x_{i+1}+\Delta x} ax^2 + bx + c\,dx = \frac{\Delta x}{3}(f(x_i) + 4f(x_{i+1}) + f(x_{i+2})).$$

Now the sum of the areas under all parabolas is

$$\begin{aligned}&\frac{\Delta x}{3}(f(x_0) + 4f(x_1) + f(x_2) + f(x_2) + 4f(x_3) + f(x_4) + \cdots + f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)) =\\ &\quad \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)).\end{aligned}$$

This is just slightly more complicated than the formula for trapezoids; we need to remember the alternating 2 and 4 coefficients, and that the interval must be divided into an *even* number of subintervals. This approximation technique is referred to as **Simpson's Rule**.

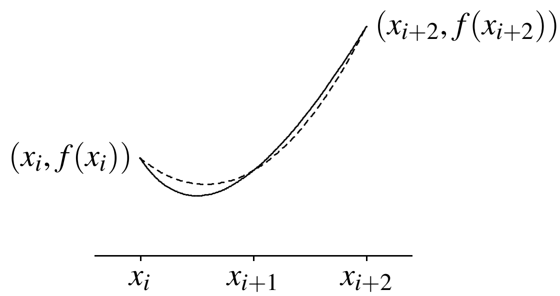


Figure 3.18: A parabola (dashed) approximating a curve (solid).

As with the trapezoid method, this is useful only with an error bound:

Theorem 3.76: Error for Simpson's Approximation

Suppose f has a fourth derivative $f^{(4)}$ everywhere on the interval $[a,b]$, and $|f^{(4)}(x)| \leq M$ for all x in the interval. With $\Delta x = (b-a)/n$, an error bound for Simpson's approximation is

$$E(\Delta x) = \frac{b-a}{180}M(\Delta x)^4 = \frac{(b-a)^5}{180n^4}M.$$

Example 3.77: Approximate an Integral With Parabolas

Let us again approximate $\int_0^1 e^{-x^2}\,dx$ to two decimal places.

Solution. The fourth derivative of $f = e^{-x^2}$ is $(16x^4 - 48x^2 + 12)e^{-x^2}$; on $[0,1]$ this is at most 12 in absolute value. We begin by estimating the number of subintervals we are likely to need. To get two decimal places of accuracy, we will certainly need $E(\Delta x) < 0.005$, but taking a cue from our earlier example, let's require $E(\Delta x) < 0.001$:

$$\begin{aligned} \frac{1}{180}(12)\frac{1}{n^4} &< 0.001 \\ \frac{200}{3} &< n^4 \\ 2.86 \approx \sqrt[4]{\frac{200}{3}} &< n \end{aligned}$$

So we try $n = 4$, since we need an even number of subintervals. Then the error bound is $12/180/4^4 < 0.0003$ and the approximation is

$$(f(0) + 4f(1/4) + 2f(1/2) + 4f(3/4) + f(1))\frac{1}{3 \cdot 4} \approx 0.746855.$$

So the true value of the integral is between $0.746855 - 0.0003 = 0.746555$ and $0.746855 + 0.0003 = 0.7471555$, both of which round to 0.75. ♣

Exercises for 3.6

In the following problems, compute the trapezoid and Simpson approximations using 4 subintervals, and compute the error bound for each. (Finding the maximum values of the second and fourth derivatives can be challenging for some of these; you may use a graphing calculator or computer software to estimate the maximum values.)

Exercise 3.6.1 $\int_1^3 x\,dx$

Exercise 3.6.2 $\int_0^3 x^2\,dx$

Exercise 3.6.3 $\int_2^4 x^3\,dx$

Exercise 3.6.4 $\int_1^3 \frac{1}{x}\,dx$

Exercise 3.6.5 $\int_1^2 \frac{1}{1+x^2}\,dx$

Exercise 3.6.6 $\int_0^1 x\sqrt{1+x}\,dx$

Exercise 3.6.7 $\int_1^5 \frac{x}{1+x}\,dx$

Exercise 3.6.8 $\int_0^1 \sqrt{x^3+1}\,dx$

Exercise 3.6.9 $\int_0^1 \sqrt{x^4+1}\,dx$

Exercise 3.6.10 $\int_1^4 \sqrt{1+1/x}\,dx$

Exercise 3.6.11 *Using Simpson's rule on a parabola $f(x)$, even with just two subintervals, gives the exact value of the integral, because the parabolas used to approximate f will be f itself. Remarkably, Simpson's rule also computes the integral of a cubic function $f(x) = ax^3+bx^2+cx+d$ exactly. Show this is true by showing that*

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{x_2-x_0}{3\cdot 2}(f(x_0)+4f((x_0+x_2)/2)+f(x_2)).$$

This does require a bit of messy algebra, so you may prefer to use Sage.

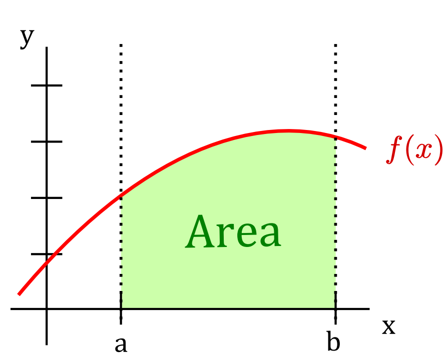
3.7 Improper Integrals

Recall that the Fundamental Theorem of Calculus says that if f is a **continuous** function on the closed interval $[a,b]$, then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a),$$

where F is any antiderivative of f.

Both the **continuity** condition and **closed interval** must hold to use the Fundamental Theorem of Calculus, and in this case, $\int_a^b f(x)\,dx$ represents the net area under $f(x)$ from a to b:



We begin with an example where blindly applying the Fundamental Theorem of Calculus can give an incorrect result.

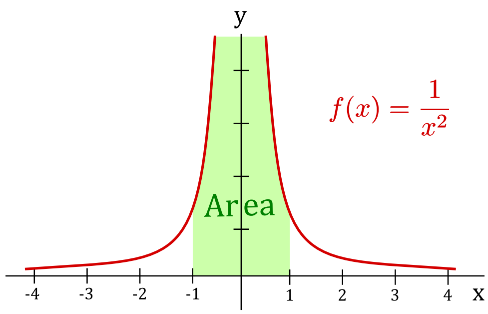
Example 3.78: Using FTC

Explain why $\int_{-1}^{1} \frac{1}{x^2}\,dx$ *is not equal to* -2.

Solution. Here is how one might proceed:

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \int_{-1}^{1} x^{-2}\,dx = -x^{-1}\Big|_{-1}^{1} = -\frac{1}{x}\Big|_{-1}^{1} = \left(-\frac{1}{1}\right) - \left(-\frac{1}{(-1)}\right) = -2$$

However, the above answer is **WRONG!** Since $f(x) = 1/x^2$ is not continuous on $[-1,\ 1]$, we cannot directly apply the Fundamental Theorem of Calculus. Intuitively, we can see why -2 is not the correct answer by looking at the graph of $f(x) = 1/x^2$ on $[-1,\ 1]$. The shaded area appears to grow without bound as seen in the figure below.



♣

Formalizing this example leads to the concept of an improper integral. There are two ways to extend the Fundamental Theorem of Calculus. One is to use an **infinite interval**, i.e., $[a,\infty)$, $(-\infty,b]$ or $(-\infty,\infty)$. The second is to allow the interval $[a,b]$ to contain an infinite **discontinuity** of $f(x)$. In either case, the integral is called an **improper integral**. One of the most important applications of this concept is probability distributions.

To compute improper integrals, we use the concept of limits along with the Fundamental Theorem of Calculus.

Definition 3.79: Definitions for Improper Integrals

If $f(x)$ *is continuous on* $[a,\infty)$, *then the improper integral of* f *over* $[a,\infty)$ *is:*

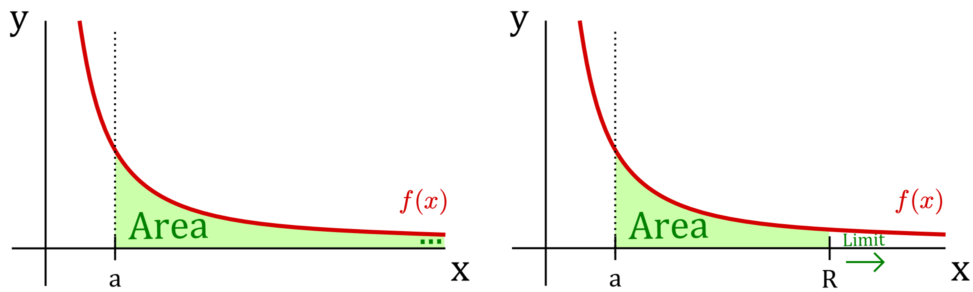
$$\int_{a}^{\infty} f(x)\,dx := \lim_{R\to\infty} \int_{a}^{R} f(x)\,dx.$$

If $f(x)$ *is continuous on* $(-\infty,b]$, *then the improper integral of* f *over* $(-\infty,b]$ *is:*

$$\int_{-\infty}^{b} f(x)\,dx := \lim_{R\to-\infty} \int_{R}^{b} f(x)\,dx.$$

If the limit exists and is a finite number, we say the improper integral **converges**. Otherwise, we say the improper integral **diverges**.

To get an intuitive (though not completely correct) interpretation of improper integrals, we attempt to analyze $\int_a^{\infty} f(x)\,dx$ graphically. Here assume $f(x)$ is continuous on $[a,\infty)$:



We let R be a fixed number in $[a,\infty)$. Then by taking the limit as R approaches ∞, we get the improper integral:

$$\int_a^{\infty} f(x)\,dx := \lim_{R\to\infty} \int_a^{R} f(x)\,dx.$$

We can then apply the Fundamental Theorem of Calculus to the last integral as $f(x)$ is continuous on the closed interval $[a,R]$.

We next define the improper integral for the interval $(-\infty,\ \infty)$.

Definition 3.80: Definitions for Improper Integrals

If both $\int_{-\infty}^{a} f(x)\,dx$ and $\int_a^{\infty} f(x)\,dx$ are convergent, then the improper integral of f over $(-\infty,\infty)$ is:

$$\int_{-\infty}^{\infty} f(x)\,dx := \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$$

The above definition requires **both** of the integrals

$$\int_{-\infty}^{a} f(x)\,dx \qquad \text{and} \qquad \int_a^{\infty} f(x)\,dx$$

to be convergent for $\int_{-\infty}^{\infty} f(x)\,dx$ to also be convergent. If **either** of $\int_{-\infty}^{a} f(x)\,dx$ or $\int_a^{\infty} f(x)\,dx$ is divergent, then so is $\int_{-\infty}^{\infty} f(x)\,dx$.

Example 3.81: Improper Integral

Determine whether $\int_1^{\infty} \frac{1}{x}\,dx$ is convergent or divergent.

Solution. Using the definition for improper integrals we write this as:

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{R\to\infty} \int_1^{R} \frac{1}{x}\,dx = \lim_{R\to\infty} \ln|x|\Big|_1^R = \lim_{R\to\infty} \ln|R| - \ln|1| = \lim_{R\to\infty} \ln|R| = +\infty$$

Therefore, the integral is **divergent**. ♣

Example 3.82: Improper Integral

Determine whether $\int_{-\infty}^{\infty} x\sin(x^2)\,dx$ *is convergent or divergent.*

Solution. We must compute both $\int_{0}^{\infty} x\sin(x^2)\,dx$ and $\int_{-\infty}^{0} x\sin(x^2)\,dx$. Note that we don't have to split the integral up at 0, any finite value a will work. First we compute the indefinite integral. Let $u = x^2$, then $du = 2x\,dx$ and hence,

$$\int x\sin(x^2)\,dx = \frac{1}{2}\int \sin(u)\,du = -\frac{1}{2}\cos(x^2) + C$$

Using the definition of improper integral gives:

$$\int_0^{\infty} x\sin(x^2)\,dx = \lim_{R\to\infty}\int_0^R x\sin(x^2)\,dx = \lim_{R\to\infty}\left[-\frac{1}{2}\cos(x^2)\right]\Bigg|_0^R = -\frac{1}{2}\lim_{R\to\infty}\cos(R^2) + \frac{1}{2}$$

This limit does not exist since $\cos x$ **oscillates** between -1 and $+1$. In particular, $\cos x$ does not approach any particular value as x gets larger and larger. Thus, $\int_0^{\infty} x\sin(x^2)\,dx$ diverges, and hence, $\int_{-\infty}^{\infty} x\sin(x^2)\,dx$ diverges. ♣

When there is a discontinuity in $[a,b]$ or at an endpoint, then the improper integral is as follows.

Definition 3.83: Definitions for Improper Integrals

If $f(x)$ *is continuous on* $(a,b]$, *then the improper integral of* f *over* $(a,b]$ *is:*

$$\int_a^b f(x)\,dx := \lim_{R\to a^+}\int_R^b f(x)\,dx.$$

If $f(x)$ *is continuous on* $[a,b)$, *then the improper integral of* f *over* $[a,b)$ *is:*

$$\int_a^b f(x)\,dx := \lim_{R\to b^-}\int_a^R f(x)\,dx.$$

If the limit above exists and is a finite number, we say the improper integral **converges**. Otherwise, we say the improper integral **diverges**.

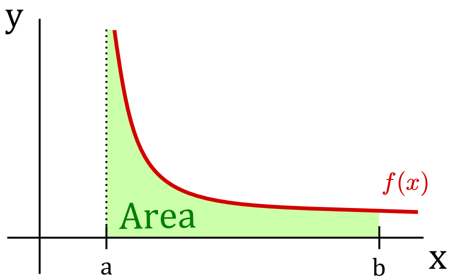
When there is a discontinuity in the interior of $[a,b]$, we use the following definition.

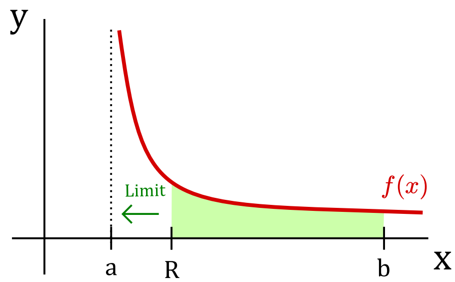
Definition 3.84: Definitions for Improper Integrals

If f *has a discontinuity at* $x = c$ *where* $c \in [a,b]$, *and both both* $\int_a^c f(x)\,dx$ *and* $\int_c^b f(x)\,dx$ *are convergent, then* f *over* $[a,b]$ *is:*

$$\int_a^b f(x)\,dx := \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Again, we can get an intuitive sense of this concept by analyzing $\int_a^b f(x)\,dx$ graphically. Here assume $f(x)$ is continuous on $(a,b]$ but discontinuous at $x = a$:





We let R be a fixed number in (a,b). Then by taking the limit as R approaches a from the **right**, we get the improper integral:

$$\int_a^b f(x)\,dx := \lim_{R\to a^+} \int_R^b f(x)\,dx.$$

Now we can apply FTC to the last integral as $f(x)$ is continuous on $[R,b]$.

Example 3.85: A Divergent Integral

Determine if $\int_{-1}^{1} \frac{1}{x^2}\,dx$ *is convergent or divergent.*

Solution. The function $f(x) = 1/x^2$ has a discontinuity at $x = 0$, which lies in $[-1,1]$. We must compute $\int_{-1}^{0} \frac{1}{x^2}\,dx$ and $\int_0^1 \frac{1}{x^2}\,dx$. Let's start with $\int_0^1 \frac{1}{x^2}\,dx$:

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{R\to 0^+} \int_R^1 \frac{1}{x^2}\,dx = \lim_{R\to 0^+} -\frac{1}{x}\bigg|_R^1 = -1 + \lim_{R\to 0^+} \frac{1}{R}$$

which diverges to $+\infty$. Therefore, $\int_{-1}^{1} \frac{1}{x^2}\,dx$ is **divergent** since one of $\int_{-1}^{0} \frac{1}{x^2}\,dx$ and $\int_0^1 \frac{1}{x^2}\,dx$ is divergent. ♣

Example 3.86: Integral of the Logarithm

Determine if $\int_0^1 \ln x\,dx$ *is convergent or divergent. Evaluate it if it is convergent.*

Solution. Note that $f(x) = \ln x$ is discontinuous at the endpoint $x = 0$. We first use integration by parts to compute $\int \ln x\,dx$. We let $u = \ln x$ and $dv = dx$. Then $du = (1/x)dx$, $v = x$, giving:

$$\begin{aligned}\int \ln x\,dx &= x\ln x - \int x\cdot\frac{1}{x}\,dx\\ &= x\ln x - \int 1\,dx\\ &= x\ln x - x + C\end{aligned}$$

Now using the definition of improper integral for $\int_0^1 \ln x\, dx$:

$$\int_0^1 \ln x\, dx = \lim_{R\to 0^+} \int_R^1 \ln x\, dx = \lim_{R\to 0^+} (x\ln x - x)\Big|_R^1 = -1 - \lim_{R\to 0^+}(R\ln R) + \lim_{R\to 0^+} R$$

Note that $\lim_{R\to 0^+} R = 0$. We next compute $\lim_{R\to 0^+}(R\ln R)$. First, we rewrite the expression as follows:

$$\lim_{x\to 0^+}(R\ln R) = \lim_{R\to 0^+} \frac{\ln R}{1/R}.$$

Now the limit is of the indeterminate type $(-\infty)/(\infty)$ and l'Hôpital's Rule can be applied.

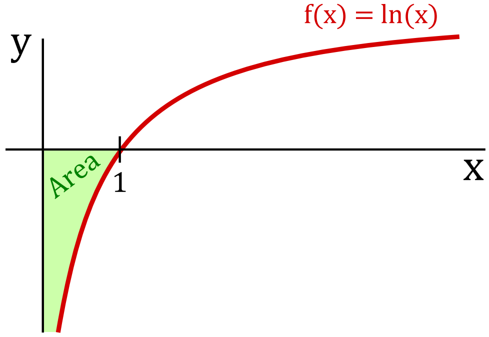
$$\lim_{R\to 0^+}(R\ln R) = \lim_{R\to 0^+} \frac{\ln R}{1/R} = \lim_{R\to 0^+} \frac{1/R}{-1/R^2} = \lim_{R\to 0^+} -\frac{R^2}{R} = \lim_{R\to 0^+}(-R) = 0$$

Thus, $\lim_{R\to 0^+}(R\ln R) = 0$. Thus

$$\int_0^1 \ln x\, dx = -1,$$

and the integral is convergent to -1.

Graphically, one might interpret this to mean that the net area under $\ln x$ on $[0,1]$ is -1 (the area in this case lies below the x-axis).



♣

Example 3.87: Integral of a Square Root

Determine if $\int_0^4 \frac{dx}{\sqrt{4-x}}$ *is convergent or divergent. Evaluate it if it is convergent.*

Solution. Note that $\frac{1}{\sqrt{4-x}}$ is discontinuous at the endpoint $x = 4$. We use a u-substitution to compute $\int \frac{dx}{\sqrt{4-x}}$. We let $u = 4-x$, then $du = -dx$, giving:

$$\begin{aligned}\int \frac{dx}{\sqrt{4-x}} &= \int -\frac{du}{u^{1/2}} \\ &= \int -u^{-1/2}\, du \\ &= -2(u)^{1/2} + C \\ &= -2\sqrt{4-x} + C\end{aligned}$$

Now using the definition of improper integrals for $\int_0^4 \frac{dx}{\sqrt{4-x}}$:

$$\int_0^4 \frac{dx}{\sqrt{4-x}} = \lim_{R\to 4^-} \left(-2\sqrt{4-x}\right)\Big|_0^R = \lim_{R\to 4^-} -2\sqrt{4-R}+2\sqrt{4} = 4$$

♣

Example 3.88: Improper Integral

Determine if $\int_1^2 \frac{dx}{(x-1)^{1/3}}$ *is convergent or divergent. Evaluate it if it is convergent.*

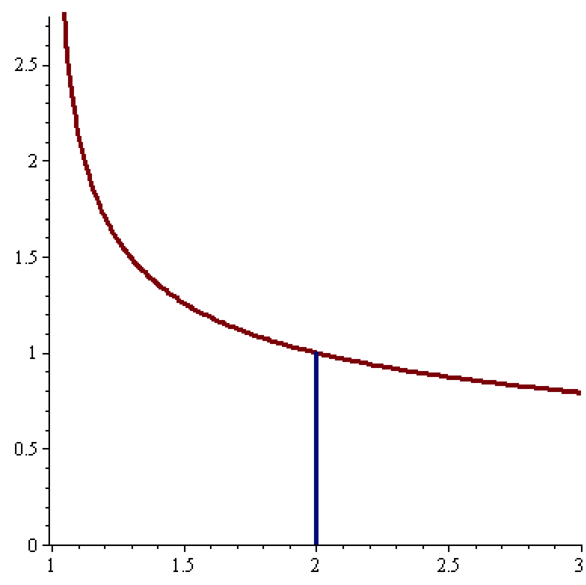
Solution. Note that $f(x) = \frac{1}{(x-1)^{1/3}}$ is discontinuous at the endpoint $x = 1$. We first use substitution to find $\int \frac{dx}{(x-1)^{1/3}}$. We let $u = x-1$. Then $du = dx$, giving

$$\int \frac{dx}{(x-1)^{1/3}} = \int \frac{du}{u^{1/3}} = \int u^{-1/3}du = \frac{3}{2}u^{2/3}+C = \frac{3}{2}(x-1)^{2/3}+C.$$

Now using the definition of improper integral for $\int_1^2 \frac{dx}{(x-1)^{1/3}}$:

$$\int_1^2 \frac{dx}{(x-1)^{1/3}} = \lim_{R\to 1^+} \int_R^2 \frac{dx}{(x-1)^{1/3}} = \lim_{R\to 1^+} \frac{3}{2}(x-1)^{2/3}\Big|_R^2 = \frac{3}{2} - \lim_{R\to 1^+} \frac{3}{2}(R-1)^{2/3} = \frac{3}{2},$$

and the integral is convergent to $\frac{3}{2}$. Graphically, one might interpret this to mean that the net area under $\frac{1}{(x-1)^{1/3}}$ on $[1,2]$ is $\frac{3}{2}$.



♣

The following test allows us to determine convergence/divergence information about improper integrals that are hard to compute by comparing them to easier ones. We state the test for $[a,\infty)$, but similar versions hold for the other improper integrals.

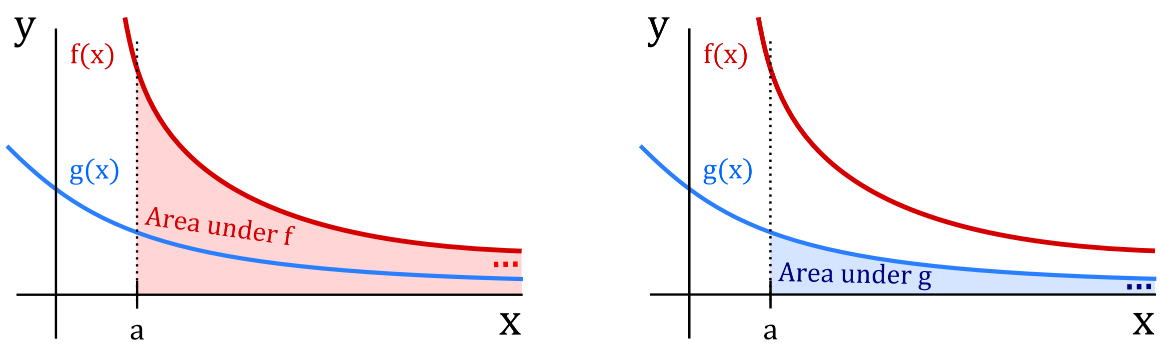
The Comparison Test

Assume that $f(x) \geq g(x) \geq 0$ for $x \geq a$.

(i) If $\int_a^\infty f(x)\,dx$ **converges**, then $\int_a^\infty g(x)\,dx$ also **converges**.

(ii) If $\int_a^\infty g(x)\,dx$ **diverges**, then $\int_a^\infty f(x)\,dx$ also **diverges**.

Informally, (i) says that if $f(x)$ is larger than $g(x)$, and the area under $f(x)$ is finite (converges), then the area under $g(x)$ must also be finite (converges). Informally, (ii) says that if $f(x)$ is larger than $g(x)$, and the area under $g(x)$ is infinite (diverges), then the area under $f(x)$ must also be infinite (diverges).



Example 3.89: Comparison Test

Show that $\int_2^\infty \frac{\cos^2 x}{x^2}\,dx$ *converges.*

Solution. We use the comparison test to show that it converges. Note that $0 \leq \cos^2 x \leq 1$ and hence

$$0 \leq \frac{\cos^2 x}{x^2} \leq \frac{1}{x^2}.$$

Thus, taking $f(x) = 1/x^2$ and $g(x) = \cos^2 x/x^2$ we have $f(x) \geq g(x) \geq 0$. One can easily see that $\int_2^\infty \frac{1}{x^2}\,dx$ converges. Therefore, $\int_2^\infty \frac{\cos^2 x}{x^2}\,dx$ also converges. ♣

Exercises for Section 3.7

Exercise 3.7.1 *Determine whether* $\int_1^{\infty} \frac{1}{x^2}\, dx$ *is convergent or divergent.*

Exercise 3.7.2 *Determine whether* $\int_e^{\infty} \frac{1}{x\sqrt{\ln x}}\, dx$ *is convergent or divergent.*

Exercise 3.7.3 *Evaluate the improper integral* $\int_0^{\infty} e^{-3x}\, dx$.

Exercise 3.7.4 *Determine if* $\int_1^{e} \frac{1}{x(\ln x)^2}\, dx$ *is convergent or divergent. Evaluate it if it is convergent.*

Exercise 3.7.5 *Show that* $\int_0^{\infty} e^{-x} \sin^2\left(\frac{\pi x}{2}\right) dx$ *converges.*

Exercise 3.7.6 *Evaluate* $\int_{-\infty}^{\infty} \frac{1}{x^2+1}\, dx$ *and* $\int_{-\infty}^{\infty} \frac{x}{x^2+1}\, dx$.

Exercise 3.7.7 *Determine whether the following improper integrals are convergent or divergent. Evaluate those that are convergent.*

(a) $\int_0^{\infty} \frac{1}{x^2+1}\, dx$

(b) $\int_0^{\infty} \frac{x}{x^2+1}\, dx$

(c) $\int_0^{\infty} e^{-x}(\cos x + \sin x)\, dx$. *[Hint: What is the derivative of* $-e^{-x}\cos x$*?]*

(d) $\int_0^{\pi/2} \sec^2 x\, dx$

(e) $\int_0^{4} \frac{1}{(4-x)^{2/5}}\, dx$

Exercise 3.7.8 *Prove that the integral* $\int_1^{\infty} \frac{1}{x^p}\, dx$ *is convergent if* $p > 1$ *and divergent if* $0 < p \leq 1$.

Exercise 3.7.9 *Suppose that* $p > 0$. *Find all values of* p *for which* $\int_0^{1} \frac{1}{x^p}\, dx$ *converges.*

Exercise 3.7.10 *Show that* $\int_1^{\infty} \frac{\sin^2 x}{x(\sqrt{x}+1)}\, dx$ *converges.*

Evaluate the given improper integral.

Exercise 3.7.11 $\int_0^{\infty} e^{5-2x}\, dx$

Exercise 3.7.12 $\int_1^{\infty} \frac{1}{x^3}\, dx$

Exercise 3.7.13 $\int_1^\infty x^{-4}\,dx$

Exercise 3.7.14 $\int_{-\infty}^\infty \frac{1}{x^2+9}\,dx$

Exercise 3.7.15 $\int_{-\infty}^0 2^x\,dx$

Exercise 3.7.16 $\int_{-\infty}^0 \left(\frac{1}{2}\right)^x\,dx$

Exercise 3.7.17 $\int_{-\infty}^\infty \frac{x}{x^2+1}\,dx$

Exercise 3.7.18 $\int_{-\infty}^\infty \frac{x}{x^2+4}\,dx$

Exercise 3.7.19 $\int_2^\infty \frac{1}{(x-1)^2}\,dx$

Exercise 3.7.20 $\int_1^2 \frac{1}{(x-1)^2}\,dx$

Exercise 3.7.21 $\int_2^\infty \frac{1}{x-1}\,dx$

Exercise 3.7.22 $\int_1^2 \frac{1}{x-1}\,dx$

Exercise 3.7.23 $\int_{-1}^1 \frac{1}{x}\,dx$

Exercise 3.7.24 $\int_1^3 \frac{1}{x-2}\,dx$

Exercise 3.7.25 $\int_0^\pi \sec^2 x\,dx$

Exercise 3.7.26 $\int_{-2}^1 \frac{1}{\sqrt{|x|}}\,dx$

Exercise 3.7.27 $\int_0^\infty xe^{-x}\,dx$

Exercise 3.7.28 $\int_0^\infty xe^{-x^2}\,dx$

Exercise 3.7.29 $\int_{-\infty}^\infty xe^{-x^2}\,dx$

Exercise 3.7.30 $\int_{-\infty}^\infty \frac{1}{e^x+e^{-x}}\,dx$

Exercise 3.7.31 $\int_0^1 x\ln x\,dx$

Exercise 3.7.32 $\int_1^\infty \frac{\ln x}{x}\,dx$

Exercise 3.7.33 $\int_0^1 \ln x\,dx$

Exercise 3.7.34 $\int_1^\infty \frac{\ln x}{x^2}\,dx$

Exercise 3.7.35 $\int_1^\infty \frac{\ln x}{\sqrt{x}}\,dx$

Exercise 3.7.36 $\int_0^\infty e^{-x}\sin x\,dx$

Exercise 3.7.37 $\int_0^\infty e^{-x}\cos x\,dx$

3.8 Additional Exercises

These problems require the techniques of this chapter, and are in no particular order. Some problems may be done in more than one way.

Exercise 3.8.1 $\int (t+4)^3\,dt$

Exercise 3.8.2 $\int t(t^2-9)^{3/2}\,dt$

Exercise 3.8.3 $\int (e^{t^2}+16)te^{t^2}\,dt$

Exercise 3.8.4 $\int \sin t\cos 2t\,dt$

Exercise 3.8.5 $\int \tan t\sec^2 t\,dt$

Exercise 3.8.6 $\int \frac{2t+1}{t^2+t+3}\,dt$

Exercise 3.8.7 $\int \frac{1}{t(t^2-4)}\,dt$

Exercise 3.8.8 $\int \frac{1}{(25-t^2)^{3/2}}\,dt$

Exercise 3.8.9 $\int \frac{\cos 3t}{\sqrt{\sin 3t}}\,dt$

Exercise 3.8.10 $\int t\sec^2 t\,dt$

Exercise 3.8.11 $\int \frac{e^t}{\sqrt{e^t+1}}\,dt$

Exercise 3.8.12 $\int \cos^4 t\,dt$

Exercise 3.8.13 $\int \frac{1}{t^2+3t}\,dt$

Exercise 3.8.14 $\int \frac{1}{t^2\sqrt{1+t^2}}\,dt$

Exercise 3.8.15 $\int \frac{\sec^2 t}{(1+\tan t)^3}\,dt$

Exercise 3.8.16 $\int t^3\sqrt{t^2+1}\,dt$

Exercise 3.8.17 $\int e^t\sin t\,dt$

Exercise 3.8.18 $\int (t^{3/2}+47)^3\sqrt{t}\,dt$

Exercise 3.8.19 $\int \frac{t^3}{(2-t^2)^{5/2}}\,dt$

Exercise 3.8.20 $\int \frac{1}{t(9+4t^2)}\,dt$

Exercise 3.8.21 $\int \frac{\arctan 2t}{1+4t^2}\,dt$

Exercise 3.8.22 $\int \frac{t}{t^2+2t-3}\,dt$

Exercise 3.8.23 $\int \sin^3 t\cos^4 t\,dt$

Exercise 3.8.24 $\int \frac{1}{t^2-6t+9}\,dt$

Exercise 3.8.25 $\int \frac{1}{t(\ln t)^2}\,dt$

Exercise 3.8.26 $\int t(\ln t)^2\,dt$

Exercise 3.8.27 $\int t^3e^t\,dt$

Exercise 3.8.28 $\int \frac{t+1}{t^2+t-1}\,dt$

Unit 4: Applications of Integration

4.1 Volume

Now that we have seen how to compute certain areas by using integration; we will now look into how some volumes may also be computed by evaluating an integral. Generally, the volumes that we can compute this way have cross-sections that are easy to describe.

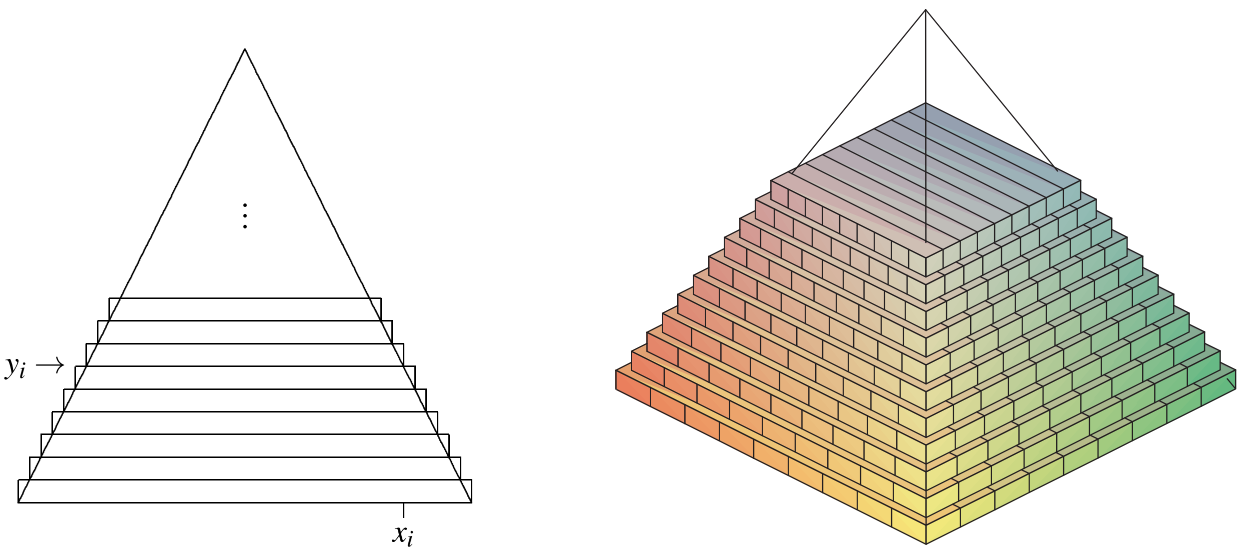


Figure 4.19: Volume of a pyramid approximated by rectangular prisms.

Example 4.90: Volume of a Pyramid

Find the volume of a pyramid with a square base that is 20 meters tall and 20 meters on a side at the base.

Solution. As with most of our applications of integration, we begin by asking how we might approximate the volume. Since we can easily compute the volume of a rectangular prism (that is, a "box"), we will use some boxes to approximate the volume of the pyramid, as shown in figure 4.19: On the left is a cross-sectional view, on the right is a 3D view of part of the pyramid with some of the boxes used to approximate the volume.

Each box has volume of the form $(2x_i)(2x_i)\Delta y$. Unfortunately, there are two variables here; fortunately, we can write x in terms of y: From the cross-sectional view we see that a height of 20 is achieved at the midpoint of the base. We will also position the cross-sectional view symmetrically about the y-axis. Thus at $x = 0$, $y = 20$, and we have a slope of $m = -2$. So

$$y = -2x + b$$

$$20 = -2(0) + b$$
$$20 = b.$$

Therefore, $y = 20 - 2x$, and in the terms of x: $x = 10 - y/2$ or $x_i = 10 - y_i/2$. Then the total volume is approximately

$$\sum_{i=0}^{n-1} 4(10 - y_i/2)^2 \Delta y$$

and in the limit we get the volume as the value of an integral:

$$\int_0^{20} 4(10 - y/2)^2\,dy = \int_0^{20} (20 - y)^2\,dy = -\frac{(20-y)^3}{3}\Bigg|_0^{20} = -\frac{0^3}{3} - -\frac{20^3}{3} = \frac{8000}{3}.$$

As you may know, the volume of a pyramid is $(1/3)(\text{height})(\text{area of base}) = (1/3)(20)(400)$, which agrees with our answer. ♣

Example 4.91: Volume of an Object

The base of a solid is the region between $f(x) = x^2 - 1$ and $g(x) = -x^2 + 1$, and its cross-sections perpendicular to the x-axis are equilateral triangles, as indicated in Figure 4.20. The solid has been truncated to show a triangular cross-section above $x = 1/2$. Find the volume of the solid.

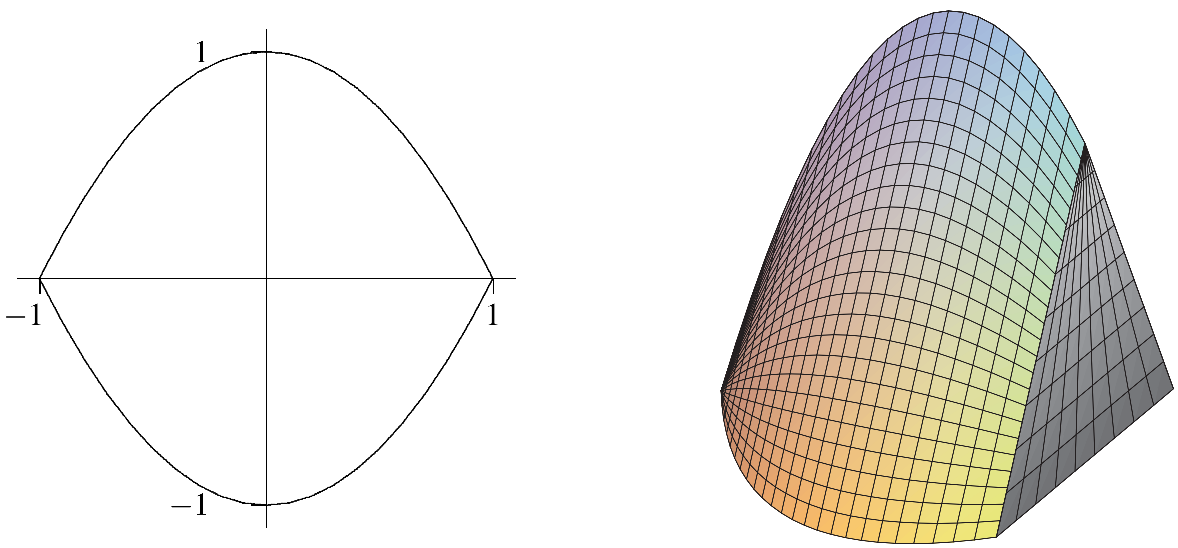


Figure 4.20: Solid with equilateral triangles as cross-sections.

Solution. A cross-section at a value x_i on the x-axis is a triangle with base $2(1 - x_i^2)$ and height $\sqrt{3}(1 - x_i^2)$, so the area of the cross-section is

$$\frac{1}{2}(\text{base})(\text{height}) = (1 - x_i^2)\sqrt{3}(1 - x_i^2),$$

and the volume of a thin "slab" is then

$$(1 - x_i^2)\sqrt{3}(1 - x_i^2)\Delta x.$$

Thus the total volume is

$$\int_{-1}^{1} \sqrt{3}(1 - x^2)^2\,dx = \frac{16}{15}\sqrt{3}.$$

♣

One easy way to get "nice" cross-sections is by rotating a plane figure around a line. For example, in Figure 4.21 we see a plane region under a curve and between two vertical lines; then the result of rotating this around the x-axis, and a typical circular cross-section is a circle.

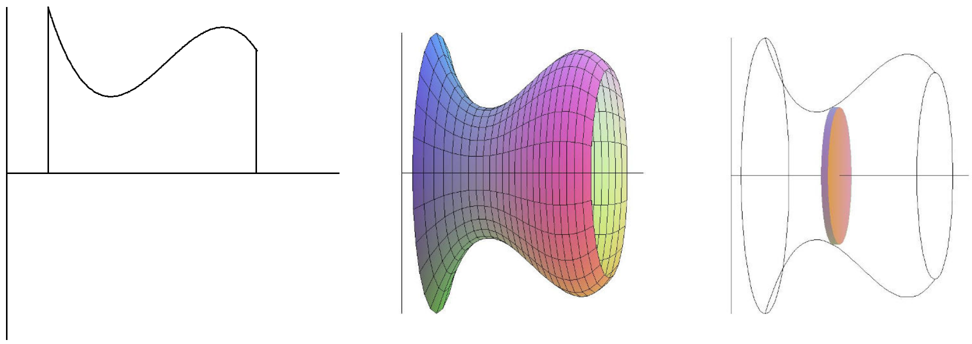


Figure 4.21: A solid of rotation.

Of course a real "slice" of this figure will not have straight sides, but we can approximate the volume of the slice by a cylinder or disk with circular top and bottom and straight sides; the volume of this disk will have the form $\pi r^2 \Delta x$. As long as we can write r in terms of x we can compute the volume by an integral.

Example 4.92: Volume of a Right Circular Cone

Find the volume of a right circular cone with base radius 10 and height 20. (A right circular cone is one with a circular base and with the tip of the cone directly over the center of the base.)

Solution. We can view this cone as produced by the rotation of the line $y = x/2$ rotated about the x-axis, as indicated in figure 4.22.

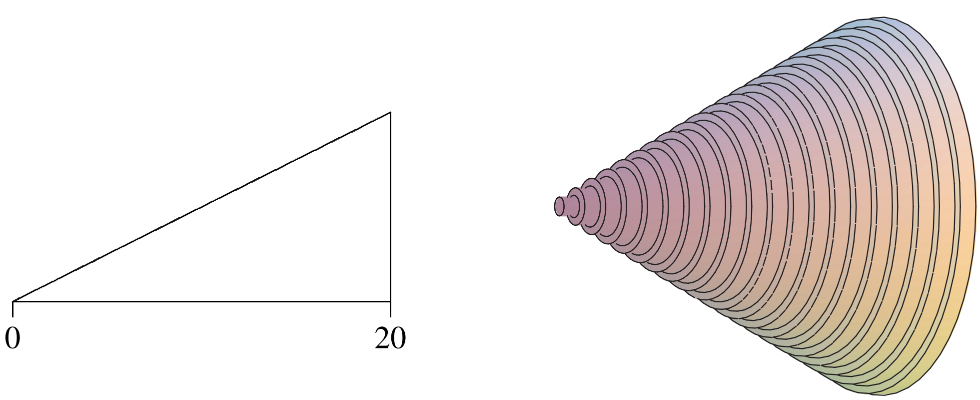


Figure 4.22: A region that generates a cone; approximating the volume by circular disks.

At a particular point on the x-axis, say x_i, the radius of the resulting cone is the y-coordinate of the corresponding point on the line, namely $y_i = x_i/2$. Thus the total volume is approximately

$$\sum_{i=0}^{n-1} \pi (x_i/2)^2 \, dx$$

and the exact volume is

$$\int_0^{20} \pi \frac{x^2}{4}\,dx = \frac{\pi}{4}\frac{20^3}{3} = \frac{2000\pi}{3}.$$

Note that we can instead do the calculation with a generic height and radius:

$$\int_0^h \pi \frac{r^2}{h^2} x^2\,dx = \frac{\pi r^2}{h^2}\frac{h^3}{3} = \frac{\pi r^2 h}{3},$$

giving us the usual formula for the volume of a cone. ♣

Example 4.93: Volume of an Object with a Hole

Find the volume of the object generated when the area between $y = x^2$ and $y = x$ is rotated around the x-axis.

Solution. This solid has a "hole" in the middle; we can compute the volume by subtracting the volume of the hole from the volume enclosed by the outer surface of the solid. In figure 4.23 we show the region that is rotated, the resulting solid with the front half cut away, the cone that forms the outer surface, the horn-shaped hole, and a cross-section perpendicular to the x-axis.

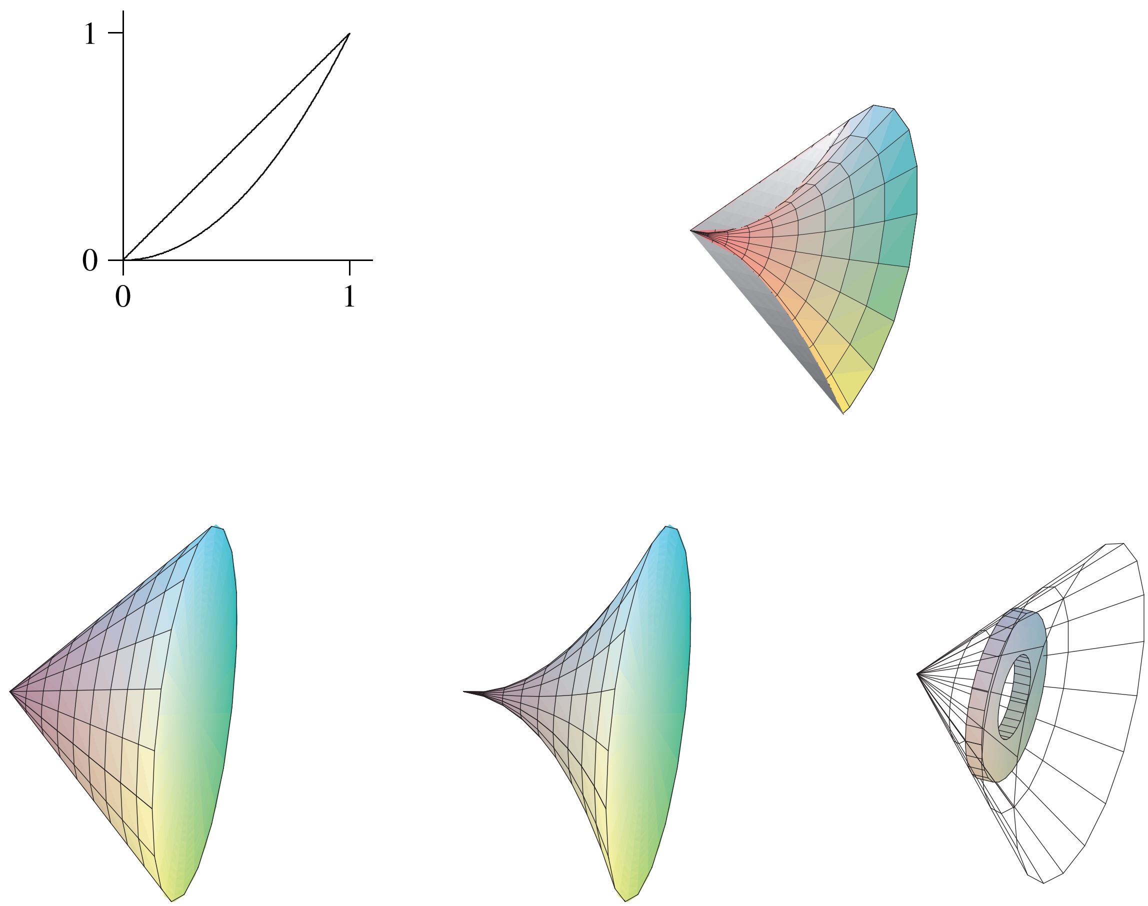


Figure 4.23: Solid with a hole, showing the outer cone and the shape to be removed to form the hole.

We have already computed the volume of a cone; in this case it is $\pi/3$. At a particular value of x, say x_i, the cross-section of the horn is a circle with radius x_i^2, so the volume of the horn is

$$\int_0^1 \pi (x^2)^2\,dx = \int_0^1 \pi x^4\,dx = \pi \frac{1}{5},$$

so the desired volume is $\pi/3 - \pi/5 = 2\pi/15$.

As with the area between curves, there is an alternate approach that computes the desired volume "all at once" by approximating the volume of the actual solid. We can approximate the volume of a slice of the solid with a washer-shaped volume, as indicated in Figure 4.23.

The volume of such a washer is the area of the face times the thickness. The thickness, as usual, is Δx, while the area of the face is the area of the outer circle minus the area of the inner circle, say $\pi R^2 - \pi r^2$, or $\pi(\text{TOP})^2 - \pi(\text{BOTTOM})^2$. In the present example, at a particular x_i, the radius R (The "TOP" function) is x_i and r (The "BOTTOM" function) x_i^2. Hence, the whole volume is

$$\int_0^1 \pi\left(\text{TOP}^2 - \text{BOTTOM}^2\right)dx = \int_0^1 \pi x^2 - \pi x^4\,dx = \pi\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Bigg|_0^1 = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15}.$$

Of course, what we have done here is exactly the same calculation as before, except we have in effect recomputed the volume of the outer cone. ♣

Suppose the region between $f(x) = x + 1$ and $g(x) = (x-1)^2$ is rotated around the y-axis; see Figure 4.24. It is possible, but inconvenient, to compute the volume of the resulting solid by the method we have used so far. The problem is that there are two "kinds" of typical rectangles: Those that go from the line to the parabola and those that touch the parabola on both ends. To compute the volume using this approach, we need to break the problem into two parts and compute two integrals:

$$\pi\int_0^1 (1+\sqrt{y})^2 - (1-\sqrt{y})^2\,dy + \pi\int_1^4 (1+\sqrt{y})^2 - (y-1)^2\,dy = \frac{8}{3}\pi + \frac{65}{6}\pi = \frac{27}{2}\pi.$$

If instead we consider a typical vertical rectangle, but still rotate around the y-axis, we get a thin "shell" instead of a thin "washer". Note that "washers" are related to the area of a circle, πr^2, whereas "shells" are related to the surface area of a cylinder, $2\pi rh$. If we add up the volume of such thin shells we will get an approximation to the true volume. What is the volume of such a shell? Consider the shell at x_i. Imagine that we cut the shell vertically in one place and "unroll" it into a thin, flat sheet, namely the surface of a cylinder. This sheet will be almost a rectangular prism that is Δx thick, $f(x_i) - g(x_i)$ (TOP−BOTTOM) tall, and $2\pi x_i$ wide. The volume will then be approximately the volume of a rectangular prism with these dimensions: $2\pi x_i(f(x_i) - g(x_i))\Delta x$. If we add these up and take the limit as usual, we get the integral

$$\int_0^3 2\pi x(f(x) - g(x))\,dx = \int_0^3 2\pi x\,(\text{TOP} - \text{BOTTOM})\,dx = \int_0^3 2\pi x(x + 1 - (x-1)^2)\,dx = \frac{27}{2}\pi.$$

Not only does this accomplish the task with only one integral, the integral is somewhat easier than those in the previous calculation. Things are not always so neat, but it is often the case that one of the two methods will be simpler than the other, so it is worth considering both before starting to do calculations.

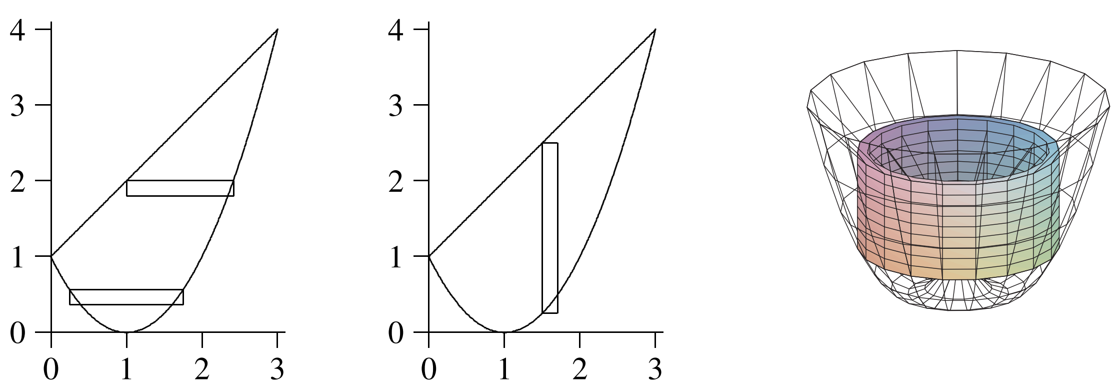


Figure 4.24: Computing volumes with "shells".

Example 4.94:

Suppose the area under $y = -x^2 + 1$ between $x = 0$ and $x = 1$ is rotated around the x-axis. Find the volume by both methods.

Solution. Using the disk method we obtain:

$$\int_0^1 \pi(1-x^2)^2\,dx = \frac{8}{15}\pi.$$

Using the shell method we obtain:

$$\int_0^1 2\pi y\sqrt{1-y}\,dy = \frac{8}{15}\pi.$$

♣

Exercises for 4.1

Exercise 4.1.1 *Verify that* $\pi\int_0^1 (1+\sqrt{y})^2-(1-\sqrt{y})^2\,dy+\pi\int_1^4 (1+\sqrt{y})^2-(y-1)^2\,dy = \frac{8}{3}\pi+\frac{65}{6}\pi = \frac{27}{2}\pi.$

Exercise 4.1.2 *Verify that* $\int_0^3 2\pi x(x+1-(x-1)^2)\,dx = \frac{27}{2}\pi.$

Exercise 4.1.3 *Verify that* $\int_0^1 \pi(1-x^2)^2\,dx = \frac{8}{15}\pi.$

Exercise 4.1.4 *Verify that* $\int_0^1 2\pi y\sqrt{1-y}\,dy = \frac{8}{15}\pi.$

Exercise 4.1.5 *Use integration to find the volume of the solid obtained by revolving the region bounded by $x+y=2$ and the x and y axes around the x-axis.*

Exercise 4.1.6 *Find the volume of the solid obtained by revolving the region bounded by $y = x - x^2$ and the x-axis around the x-axis.*

Exercise 4.1.7 *Find the volume of the solid obtained by revolving the region bounded by $y = \sqrt{\sin x}$ between $x = 0$ and $x = \pi/2$, the y-axis, and the line $y = 1$ around the x-axis.*

Exercise 4.1.8 *Let S be the region of the xy-plane bounded above by the curve $x^3y = 64$, below by the line $y = 1$, on the left by the line $x = 2$, and on the right by the line $x = 4$. Find the volume of the solid obtained by rotating S around:*

(a) the x-axis;

(b) the line $y = 1$;

(c) the y-axis; and

(d) the line $x = 2$.

Exercise 4.1.9 *The equation* $x^2/9+y^2/4=1$ *describes an ellipse. Find the volume of the solid obtained by rotating the ellipse around the x-axis and also around the y-axis. These solids are called* **ellipsoids***; one is vaguely rugby-ball shaped, one is sort of flying-saucer shaped, or perhaps squished-beach-ball-shaped.*

Figure 4.25: Ellipsoids.

Exercise 4.1.10 *Use integration to compute the volume of a sphere of radius r. You should of course get the well-known formula* $4\pi r^3/3$.

Exercise 4.1.11 *A hemispheric bowl of radius r contains water to a depth h. Find the volume of water in the bowl.*

Exercise 4.1.12 *The base of a tetrahedron (a triangular pyramid) of height h is an equilateral triangle of side s. Its cross-sections perpendicular to an altitude are equilateral triangles. Express its volume V as an integral, and find a formula for V in terms of h and s. Verify that your answer is* $(1/3)$(*area of base*)(*height*).

Exercise 4.1.13 *The base of a solid is the region between* $f(x)=\cos x$ *and* $g(x)=-\cos x$, $-\pi/2\le x\le \pi/2$, *and its cross-sections perpendicular to the x-axis are squares. Find the volume of the solid.*

4.2 Arc Length

Here is another geometric application of the integral: Find the length of a portion of a curve. As usual, we need to think about how we might approximate the length, and turn the approximation into an integral.

We already know how to compute one simple arc length, that of a line segment. If the endpoints are $P_0(x_0,y_0)$ and $P_1(x_1,y_1)$ then the length of the segment is the distance between the points, $\sqrt{(x_1-x_0)^2+(y_1-y_0)^2}$, from the Pythagorean theorem, as illustrated in Figure 4.26.

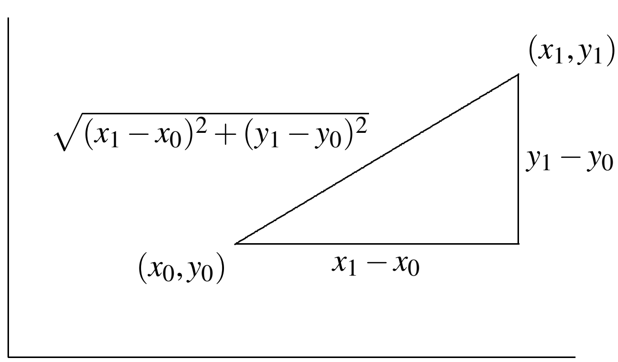


Figure 4.26: The length of a line segment.

Now if the graph of f is "nice" (say, differentiable) it appears that we can approximate the length of a portion of the curve with line segments, and that as the number of segments increases, and their lengths decrease, the sum of the lengths of the line segments will approach the true arc length; see Figure 4.27.

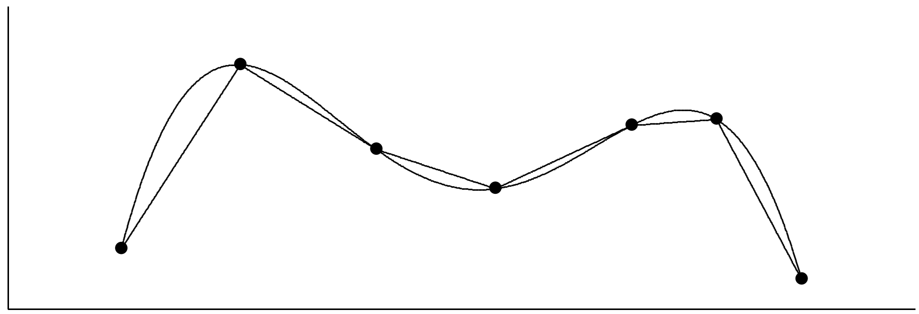


Figure 4.27: Approximating arc length with line segments.

Now we need to write a formula for the sum of the lengths of the line segments, in a form that we know becomes an integral in the limit. So we suppose we have divided the interval $[a,b]$ into n subintervals as usual, each with length $\Delta x = (b-a)/n$, and endpoints $a = x_0, x_1, x_2, \ldots, x_n = b$. The length of a typical line segment, joining $(x_i, f(x_i))$ to $(x_{i+1}, f(x_{i+1}))$, is $\sqrt{(\Delta x)^2 + (f(x_{i+1}) - f(x_i))^2}$. By the Mean Value Theorem, there is a number t_i in (x_i, x_{i+1}) such that $f'(t_i)\Delta x = f(x_{i+1}) - f(x_i)$, so the length of the line segment can be written as

$$\sqrt{(\Delta x)^2 + (f'(t_i))^2 \Delta x^2} = \sqrt{1 + (f'(t_i))^2}\,\Delta x.$$

Then arc length is:

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} \sqrt{1 + (f'(t_i))^2}\,\Delta x = \int_a^b \sqrt{1 + (f'(x))^2}\,dx.$$

Note that the sum looks a bit different than others we have encountered, because the approximation contains a t_i instead of an x_i. In the past we have always used left endpoints (namely, x_i) to get a representative value of f on $[x_i, x_{i+1}]$; now we are using a different point, but the principle is the same.

To summarize, to compute the length of a curve on the interval $[a,b]$, we compute the integral

$$\int_a^b \sqrt{1 + (f'(x))^2}\,dx.$$

Unfortunately, integrals of this form are typically difficult or impossible to compute exactly, because usually none of our methods for finding antiderivatives will work. In practice this means that the integral will usually have to be approximated.

Example 4.95: Circumference of a Circle

Let $f(x) = \sqrt{r^2 - x^2}$, the upper half circle of radius r. The length of this curve is half the circumference, namely πr. Compute this with the arc length formula.

Solution. The derivative f' is $-x/\sqrt{r^2 - x^2}$ so the integral is

$$\int_{-r}^{r} \sqrt{1 + \frac{x^2}{r^2 - x^2}}\,dx = \int_{-r}^{r} \sqrt{\frac{r^2}{r^2 - x^2}}\,dx = r\int_{-r}^{r} \sqrt{\frac{1}{r^2 - x^2}}\,dx.$$

Using a trigonometric substitution, we find the antiderivative, namely $\arcsin(x/r)$. Notice that the integral is improper at both endpoints, as the function $\sqrt{1/(r^2-x^2)}$ is undefined when $x=\pm r$. So we need to compute

$$\lim_{D\to -r^+}\int_D^0 \sqrt{\frac{1}{r^2-x^2}}\,dx + \lim_{D\to r^-}\int_0^D \sqrt{\frac{1}{r^2-x^2}}\,dx.$$

This is not difficult, and has value π, so the original integral, with the extra r in front, has value πr as expected. ♣

Exercises for 4.2

Exercise 4.2.1 *Find the arc length of* $f(x)=x^{3/2}$ *on* $[0,2]$.

Exercise 4.2.2 *Find the arc length of* $f(x)=x^2/8-\ln x$ *on* $[1,2]$.

Exercise 4.2.3 *Find the arc length of* $f(x)=(1/3)(x^2+2)^{3/2}$ *on the interval* $[0,a]$.

Exercise 4.2.4 *Find the arc length of* $f(x)=\ln(\sin x)$ *on the interval* $[\pi/4,\pi/3]$.

Exercise 4.2.5 *Let* $a>0$. *Show that the length of* $y=\cosh x$ *on* $[0,a]$ *is equal to* $\int_0^a \cosh x\,dx$.

Exercise 4.2.6 *Find the arc length of* $f(x)=\cosh x$ *on* $[0,\ln 2]$.

Exercise 4.2.7 *Set up the integral to find the arc length of* $\sin x$ *on the interval* $[0,\pi]$*; do not evaluate the integral. If you have access to appropriate software, approximate the value of the integral.*

Exercise 4.2.8 *Set up the integral to find the arc length of* $y=xe^{-x}$ *on the interval* $[2,3]$*; do not evaluate the integral. If you have access to appropriate software, approximate the value of the integral.*

Exercise 4.2.9 *Find the arc length of* $y=e^x$ *on the interval* $[0,1]$. *(This can be done exactly; it is a bit tricky and a bit long.)*

Find the arc length of the function on the given interval.

Exercise 4.2.10 $f(x)=x$ *on* $[0,1]$

Exercise 4.2.11 $f(x)=\sqrt{8}x$ *on* $[-1,1]$

Exercise 4.2.12 $f(x)=\frac{1}{3}x^{3/2}-x^{1/2}$ *on* $[0,1]$

Exercise 4.2.13 $f(x)=\frac{1}{12}x^3+\frac{1}{x}$ *on* $[1,4]$

Exercise 4.2.14 $f(x)=2x^{3/2}-\frac{1}{6}\sqrt{x}$ *on* $[0,9]$

Exercise 4.2.15 $f(x)=\cosh x$ *on* $[-\ln 2,\ln 2]$

Exercise 4.2.16 $f(x)=\frac{1}{2}\left(e^x+e^{-x}\right)$ *on* $[0,\ln 5]$

Exercise 4.2.17 $f(x)=\frac{1}{12}x^5+\frac{1}{5x^3}$ *on* $[.1,1]$

Exercise 4.2.18 $f(x)=\ln\left(\sin x\right)$ *on* $[\pi/6,\pi/2]$

Exercise 4.2.19 $f(x)=\ln\left(\cos x\right)$ *on* $[0,\pi/4]$

Set up the integral to compute the arc length of the function on the given interval. Do not evaluate the integral.

Exercise 4.2.20 $f(x) = x^2$ *on* $[0,1]$

Exercise 4.2.21 $f(x) = x^{10}$ *on* $[0,1]$

Exercise 4.2.22 $f(x) = \sqrt{x}$ *on* $[0,1]$

Exercise 4.2.23 $f(x) = \ln x$ *on* $[1,e]$

Exercise 4.2.24 $f(x) = \sqrt{1-x^2}$ *on* $[-1,1]$. *(Note: this describes the top half of a circle with radius 1.)*

Exercise 4.2.25 $f(x) = \sqrt{1-x^2/9}$ *on* $[-3,3]$. *(Note: this describes the top half of an ellipse with a major axis of length 6 and a minor axis of length 2.)*

Exercise 4.2.26 $f(x) = \dfrac{1}{x}$ *on* $[1,2]$

4.3 Surface Area

Another geometric question that arises naturally is: "What is the surface area of a volume?" For example, what is the surface area of a sphere? More advanced techniques are required to approach this question in general, but we can compute the areas of some volumes generated by revolution.

As usual, the question is: How might we approximate the surface area? For a surface obtained by rotating a curve around an axis, we can take a polygonal approximation to the curve, as in the last section, and rotate it around the same axis. This gives a surface composed of many "truncated cones"; a truncated cone is called a **frustum** of a cone. Figure 4.28 illustrates this approximation.

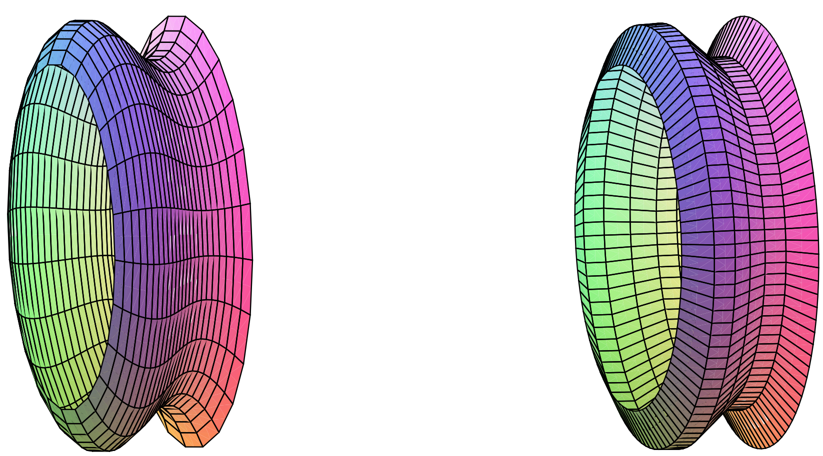


Figure 4.28: Approximating a surface (left) by portions of cones (right).

So we need to be able to compute the area of a frustum of a cone. Since the frustum can be formed by removing a small cone from the top of a larger one, we can compute the desired area if we know the surface area of a cone. Suppose a right circular cone has base radius r and slant height h. If we cut the cone from the vertex to the base circle and flatten it out, we obtain a sector of a circle with radius h and arc length $2\pi r$, as in Figure 4.29. The angle at the center, in radians, is then $2\pi r/h$, and the area of the cone is equal to the area of the sector of the circle. Let A be the area of the sector; since the area of the entire circle is πh^2, we have

$$\frac{A}{\pi h^2} = \frac{2\pi r/h}{2\pi}$$

$$A = \pi r h.$$

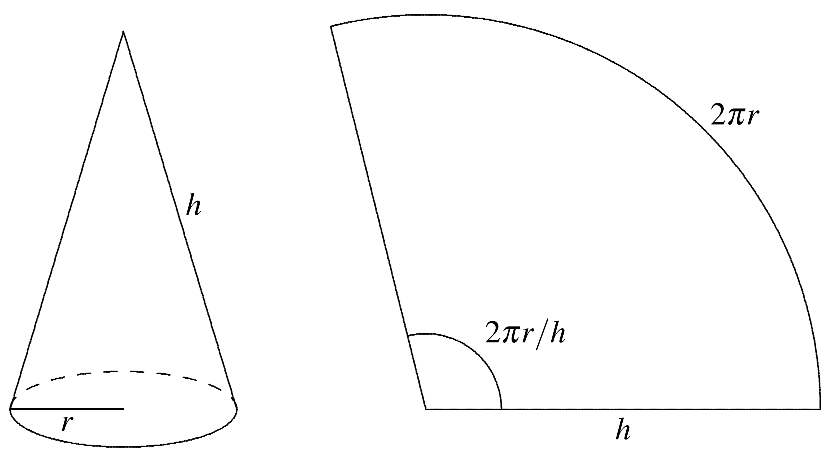


Figure 4.29: The area of a cone.

Now suppose we have a frustum of a cone with slant height h and radii r_0 and r_1, as in Figure 4.30. The area of the entire cone is $\pi r_1(h_0 + h)$, and the area of the small cone is $\pi r_0 h_0$; thus, the area of the frustum is $\pi r_1(h_0+h) - \pi r_0 h_0 = \pi((r_1 - r_0)h_0 + r_1 h)$. By similar triangles,

$$\frac{h_0}{r_0} = \frac{h_0 + h}{r_1}.$$

With a bit of algebra this becomes $(r_1 - r_0)h_0 = r_0 h$; substitution into the area gives

$$\pi((r_1 - r_0)h_0 + r_1 h) = \pi(r_0 h + r_1 h) = \pi h(r_0 + r_1) = 2\pi \frac{r_0 + r_1}{2} h = 2\pi r h.$$

The final form is particularly easy to remember, with r equal to the average of r_0 and r_1, as it is also the formula for the area of a cylinder. (Think of a cylinder of radius r and height h as the frustum of a cone of infinite height.)

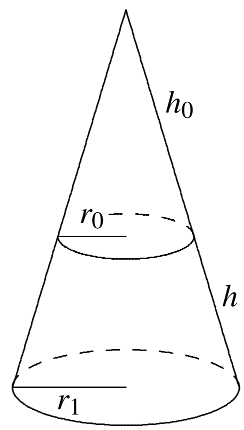


Figure 4.30: The area of a frustum.

Now we are ready to approximate the area of a surface of revolution. On one subinterval, the situation is as shown in Figure 4.31. When the line joining two points on the curve is rotated around the x-axis, it forms a frustum of a cone. The area is

$$2\pi rh = 2\pi \frac{f(x_i)+f(x_{i+1})}{2}\sqrt{1+(f'(t_i))^2}\,\Delta x.$$

Here $\sqrt{1+(f'(t_i))^2}\,\Delta x$ is the length of the line segment, as we found in the previous section. Assuming f is a continuous function, there must be some x_i^* in $[x_i, x_{i+1}]$ such that $(f(x_i)+f(x_{i+1}))/2 = f(x_i^*)$, so the approximation for the surface area is

$$\sum_{i=0}^{n-1} 2\pi f(x_i^*)\sqrt{1+(f'(t_i))^2}\,\Delta x.$$

This is not quite the sort of sum we have seen before, as it contains two different values in the interval $[x_i, x_{i+1}]$, namely x_i^* and t_i. Nevertheless, using more advanced techniques than we have available here, it turns out that

$$\lim_{n\to\infty}\sum_{i=0}^{n-1} 2\pi f(x_i^*)\sqrt{1+(f'(t_i))^2}\,\Delta x = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$

is the surface area we seek. (Roughly speaking, this is because while x_i^* and t_i are distinct values in $[x_i, x_{i+1}]$, they get closer and closer to each other as the length of the interval shrinks.)

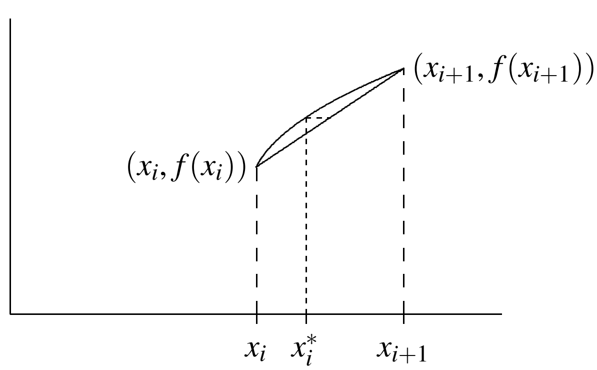


Figure 4.31: One subinterval.

Example 4.96: Surface Area of a Sphere

Compute the surface area of a sphere of radius r.

Solution. The sphere can be obtained by rotating the graph of $f(x) = \sqrt{r^2 - x^2}$ about the x-axis. The derivative f' is $-x/\sqrt{r^2 - x^2}$, so the surface area is given by

$$\begin{aligned} A &= 2\pi \int_{-r}^{r} \sqrt{r^2 - x^2}\sqrt{1 + \frac{x^2}{r^2 - x^2}}\,dx \\ &= 2\pi \int_{-r}^{r} \sqrt{r^2 - x^2}\sqrt{\frac{r^2}{r^2 - x^2}}\,dx \\ &= 2\pi \int_{-r}^{r} r\,dx = 2\pi r \int_{-r}^{r} 1\,dx = 4\pi r^2 \end{aligned}$$

♣

If the curve is rotated around the y axis, the formula is nearly identical, because the length of the line segment we use to approximate a portion of the curve doesn't change. Instead of the radius $f(x_i^*)$, we use the new radius $\bar{x}_i = (x_i + x_{i+1})/2$, and the surface area integral becomes

$$\int_a^b 2\pi x \sqrt{1 + (f'(x))^2}\,dx.$$

Example 4.97: Surface Around y-axis

Compute the area of the surface formed when $f(x) = x^2$ between 0 and 2 is rotated around the y-axis.

Solution. We compute $f'(x) = 2x$, and then

$$2\pi \int_0^2 x\sqrt{1 + 4x^2}\,dx = \frac{\pi}{6}(17^{3/2} - 1),$$

by a simple substitution. ♣

Exercises for 4.3

Exercise 4.3.1 *Compute the area of the surface formed when $f(x) = 2\sqrt{1-x}$ between -1 and 0 is rotated around the x-axis.*

Exercise 4.3.2 *Compute the surface area of example 4.97 by rotating $f(x) = \sqrt{x}$ around the x-axis.*

Exercise 4.3.3 *Compute the area of the surface formed when $f(x) = x^3$ between 1 and 3 is rotated around the x-axis.*

Exercise 4.3.4 *Compute the area of the surface formed when $f(x) = 2 + \cosh(x)$ between 0 and 1 is rotated around the x-axis.*

Exercise 4.3.5 *Consider the surface obtained by rotating the graph of $f(x) = 1/x$, $x \geq 1$, around the x-axis. This surface is called* **Gabriel's horn** *or* **Toricelli's trumpet**. *Show that Gabriel's horn has infinite surface area.*

Exercise 4.3.6 *Consider the circle $(x-2)^2 + y^2 = 1$. Sketch the surface obtained by rotating this circle about the y-axis. (The surface is called a* **torus**.*) What is the surface area?*

Exercise 4.3.7 *Consider the ellipse with equation $x^2/4 + y^2 = 1$. If the ellipse is rotated around the x-axis it forms an* **ellipsoid**. *Compute the surface area.*

Exercise 4.3.8 *Generalize the preceding result: rotate the ellipse given by $x^2/a^2 + y^2/b^2 = 1$ about the x-axis and find the surface area of the resulting ellipsoid. You should consider two cases, when $a > b$ and when $a < b$. Compare to the area of a sphere.*

4.4 Center of Mass

Suppose a beam is 10 meters long, and that there are three weights on the beam: a 10 kilogram weight 3 meters from the left end, a 5 kilogram weight 6 meters from the left end, and a 4 kilogram weight 8 meters from the left end. Where should a fulcrum be placed so that the beam balances? Let's assign a scale to the beam, from 0 at the left end to 10 at the right, so that we can denote locations on the beam simply as x coordinates; the weights are at $x = 3$, $x = 6$, and $x = 8$, as in Figure 4.32.

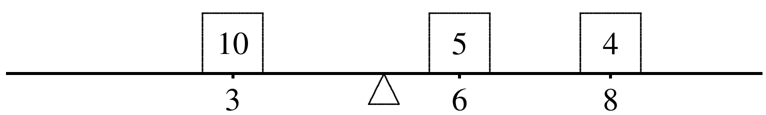


Figure 4.32: A beam with three masses.

Suppose to begin with that the fulcrum is placed at $x = 5$. What will happen? Each weight applies a force to the beam that tends to rotate it around the fulcrum; this effect is measured by a quantity called **torque**, proportional to the mass times the distance from the fulcrum. Of course, weights on different sides of the fulcrum rotate the beam in opposite directions. We can distinguish this by using a signed distance in the formula for torque. So with the fulcrum at 5, the torques induced by the three weights will be proportional to $(3-5)10 = -20$, $(6-5)5 = 5$, and $(8-5)4 = 12$. For the beam to balance, the sum of the torques must be zero; since the sum is $-20+5+12 = -3$, the beam rotates counter-clockwise, and to get the beam to balance we need to move the fulcrum to the left. To calculate exactly where the fulcrum should be, we let $\bar{x}$ denote the location of the fulcrum when the beam is in balance. The total torque on the beam is then $(3-\bar{x})10 + (6-\bar{x})5 + (8-\bar{x})4 = 92 - 19\bar{x}$. Since the beam balances at $\bar{x}$ it must be that $92 - 19\bar{x} = 0$ or $\bar{x} = 92/19 \approx 4.84$, that is, the fulcrum should be placed at $x = 92/19$ to balance the beam.

Now suppose that we have a beam with varying density—some portions of the beam contain more mass than other portions of the same size. We want to figure out where to put the fulcrum so that the beam balances.

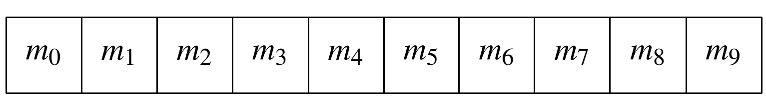


Figure 4.33: A solid beam.

Example 4.98: Balance Point of a Beam

Find the balance point of a solid beam, illustrated in Figure 4.33, assuming the beam is 10 meters long and that the density is $1+x$ *kilograms per meter at location* x *on the beam.*

Solution. To approximate the solution, we can think of the beam as a sequence of weights "on" a beam. For example, we can think of the portion of the beam between $x=0$ and $x=1$ as a weight sitting at $x=0$, the portion between $x=1$ and $x=2$ as a weight sitting at $x=1$, and so on, as indicated in Figure 4.33. We then approximate the mass of the weights by assuming that each portion of the beam has constant density. So the mass of the first weight is approximately $m_0=(1+0)1=1$ kilograms, namely, $(1+0)$ kilograms per meter times 1 meter. The second weight is $m_1=(1+1)1=2$ kilograms, and so on to the tenth weight with $m_9=(1+9)1=10$ kilograms. So in this case the total torque is

$$(0-\bar{x})m_0+(1-\bar{x})m_1+\cdots+(9-\bar{x})m_9=(0-\bar{x})1+(1-\bar{x})2+\cdots+(9-\bar{x})10.$$

If we set this to zero and solve for $\bar{x}$ we get $\bar{x}=6$. In general, if we divide the beam into n portions, the mass of weight number i will be $m_i=(1+x_i)(x_{i+1}-x_i)=(1+x_i)\Delta x$ and the torque induced by weight number i will be $(x_i-\bar{x})m_i=(x_i-\bar{x})(1+x_i)\Delta x$. The total torque is then

$$\begin{aligned}
&(x_0-\bar{x})(1+x_0)\Delta x+(x_1-\bar{x})(1+x_1)\Delta x+\cdots+(x_{n-1}-\bar{x})(1+x_{n-1})\Delta x\\
&=\sum_{i=0}^{n-1}x_i(1+x_i)\Delta x-\sum_{i=0}^{n-1}\bar{x}(1+x_i)\Delta x\\
&=\sum_{i=0}^{n-1}x_i(1+x_i)\Delta x-\bar{x}\sum_{i=0}^{n-1}(1+x_i)\Delta x.
\end{aligned}$$

If we set this equal to zero and solve for $\bar{x}$ we get an approximation to the balance point of the beam:

$$\begin{aligned}
0&=\sum_{i=0}^{n-1}x_i(1+x_i)\Delta x-\bar{x}\sum_{i=0}^{n-1}(1+x_i)\Delta x\\
\bar{x}\sum_{i=0}^{n-1}(1+x_i)\Delta x&=\sum_{i=0}^{n-1}x_i(1+x_i)\Delta x\\
\bar{x}&=\frac{\sum_{i=0}^{n-1}x_i(1+x_i)\Delta x}{\sum_{i=0}^{n-1}(1+x_i)\Delta x}.
\end{aligned}$$

The denominator of this fraction has a very familiar interpretation. Consider one term of the sum in the denominator: $(1+x_i)\Delta x$. This is the density near x_i times a short length, Δx, which in other words is approximately the mass of the beam between x_i and x_{i+1}. When we add these up we get approximately the mass of the beam.

Now each of the sums in the fraction has the right form to turn into an integral, which in turn gives us the exact value of $\bar{x}$:

$$\bar{x}=\frac{\int_0^{10}x(1+x)\,dx}{\int_0^{10}(1+x)\,dx}.$$

The numerator of this fraction is called the **moment** of the system around zero:

$$\int_0^{10}x(1+x)\,dx=\int_0^{10}x+x^2\,dx=\frac{1150}{3},$$

and the denominator is the mass of the beam:

$$\int_0^{10} (1+x)\,dx = 60,$$

and the balance point, officially called the **center of mass** is

$$\bar{x} = \frac{1150}{3}\frac{1}{60} = \frac{115}{18} \approx 6.39.$$

♣

It should be apparent that there was nothing special about the density function $\sigma(x) = 1+x$ or the length of the beam, or even that the left end of the beam is at the origin. In general, if the density of the beam is $\sigma(x)$ and the beam covers the interval $[a,b]$, the moment of the beam around zero is

$$M_0 = \int_a^b x\sigma(x)\,dx$$

and the total mass of the beam is

$$M = \int_a^b \sigma(x)\,dx$$

and the center of mass is at

$$\bar{x} = \frac{M_0}{M}.$$

Example 4.99: Center of Mass of a Beam

Suppose a beam lies on the x-axis between 20 and 30, and has density function $\sigma(x) = x-19$. Find the center of mass.

Solution. This is the same as the previous example except that the beam has been moved. Note that the density at the left end is $20-19=1$ and at the right end is $30-19=11$, as before. Hence the center of mass must be at approximately $20+6.39=26.39$. Let's see how the calculation works out.

$$M_0 = \int_{20}^{30} x(x-19)\,dx = \int_{20}^{30} x^2 - 19x\,dx = \left.\frac{x^3}{3} - \frac{19x^2}{2}\right|_{20}^{30} = \frac{4750}{3}$$

$$M = \int_{20}^{30} x - 19\,dx = \left.\frac{x^2}{2} - 19x\right|_{20}^{30} = 60$$

$$\frac{M_0}{M} = \frac{4750}{3}\frac{1}{60} = \frac{475}{18} \approx 26.39.$$

♣

Example 4.100: Centroid of a Flat Plate

Suppose a flat plate of uniform density has the shape contained by $y = x^2$, $y = 1$, and $x = 0$, in the first quadrant. Find the center of mass. (Since the density is constant, the center of mass depends only on the shape of the plate, not the density, or in other words, this is a purely geometric quantity. In such a case the center of mass is called the ***centroid****.)*

Solution. This is a two dimensional problem, but it can be solved as if it were two one dimensional problems: we need to find the x and y coordinates of the center of mass, $\bar{x}$ and $\bar{y}$, and fortunately we can do these independently. Imagine looking at the plate edge on, from below the x-axis. The plate will appear to be a beam, and the mass of a short section of the "beam", say between x_i and x_{i+1}, is the mass of a strip of the plate between x_i and x_{i+1}. See Figure 4.34 showing the plate from above and as it appears edge on.

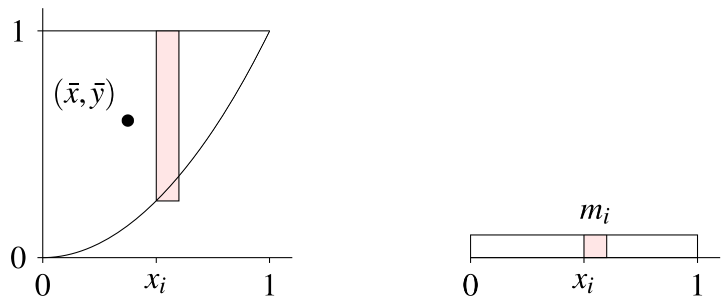


Figure 4.34: Center of mass for a two dimensional plate.

Since the plate has uniform density we may as well assume that $\sigma = 1$. Then the mass of the plate between x_i and x_{i+1} is approximately $m_i = \sigma(1-x_i^2)\Delta x = (1-x_i^2)\Delta x$. Now we can compute the moment around the y-axis:

$$M_y = \int_0^1 x(1-x^2)\,dx = \frac{1}{4}$$

and the total mass

$$M = \int_0^1 (1-x^2)\,dx = \frac{2}{3}$$

and finally

$$\bar{x} = \frac{1}{4}\frac{3}{2} = \frac{3}{8}.$$

Next we do the same thing to find $\bar{y}$. The mass of the plate between y_i and y_{i+1} is approximately $n_i = \sqrt{y}\Delta y$, so

$$M_x = \int_0^1 y\sqrt{y}\,dy = \frac{2}{5}$$

and

$$\bar{y} = \frac{2}{5}\frac{3}{2} = \frac{3}{5},$$

since the total mass M is the same. The center of mass is shown in Figure 4.34. ♣

Example 4.101: Center of Mass under Cosine

Find the center of mass of a thin, uniform plate whose shape is the region between $y = \cos x$ and the x-axis between $x = -\pi/2$ and $x = \pi/2$.

Solution. It is clear that $\bar{x} = 0$, but for practice let's compute it anyway. We will need the total mass, so we compute it first:

$$M = \int_{-\pi/2}^{\pi/2} \cos x\,dx = \sin x\Big|_{-\pi/2}^{\pi/2} = 2.$$

The moment around the y-axis is

$$M_y = \int_{-\pi/2}^{\pi/2} x\cos x\,dx = \cos x + x\sin x\Big|_{-\pi/2}^{\pi/2} = 0$$

and the moment around the x-axis is

$$M_x = \int_0^1 y\cdot 2\arccos y\,dy = y^2\arccos y - \frac{y\sqrt{1-y^2}}{2} + \frac{\arcsin y}{2}\Big|_0^1 = \frac{\pi}{4}.$$

Thus

$$\bar{x} = \frac{0}{2}, \quad \bar{y} = \frac{\pi}{8} \approx 0.393.$$

♣

Exercises for 4.4

Exercise 4.4.1 *A beam 10 meters long has density $\sigma(x) = x^2$ at distance x from the left end of the beam. Find the center of mass $\bar{x}$.*

Exercise 4.4.2 *A beam 10 meters long has density $\sigma(x) = \sin(\pi x/10)$ at distance x from the left end of the beam. Find the center of mass $\bar{x}$.*

Exercise 4.4.3 *A beam 4 meters long has density $\sigma(x) = x^3$ at distance x from the left end of the beam. Find the center of mass $\bar{x}$.*

Exercise 4.4.4 *Verify that* $\displaystyle\int 2x\arccos x\,dx = x^2\arccos x - \frac{x\sqrt{1-x^2}}{2} + \frac{\arcsin x}{2} + C.$

Exercise 4.4.5 *A thin plate lies in the region between $y = x^2$ and the x-axis between $x = 1$ and $x = 2$. Find the centroid.*

Exercise 4.4.6 *A thin plate fills the upper half of the unit circle $x^2 + y^2 = 1$. Find the centroid.*

Exercise 4.4.7 *A thin plate lies in the region contained by $y = x$ and $y = x^2$. Find the centroid.*

Exercise 4.4.8 *A thin plate lies in the region contained by $y = 4 - x^2$ and the x-axis. Find the centroid.*

Exercise 4.4.9 *A thin plate lies in the region contained by $y = x^{1/3}$ and the x-axis between $x = 0$ and $x = 1$. Find the centroid.*

Exercise 4.4.10 *A thin plate lies in the region contained by $\sqrt{x} + \sqrt{y} = 1$ and the axes in the first quadrant. Find the centroid.*

Exercise 4.4.11 *A thin plate lies in the region between the circle $x^2 + y^2 = 4$ and the circle $x^2 + y^2 = 1$, above the x-axis. Find the centroid.*

Exercise 4.4.12 *A thin plate lies in the region between the circle* $x^2+y^2=4$ *and the circle* $x^2+y^2=1$ *in the first quadrant. Find the centroid.*

Exercise 4.4.13 *A thin plate lies in the region between the circle* $x^2+y^2=25$ *and the circle* $x^2+y^2=16$ *above the* x*-axis. Find the centroid.*

Unit 5: Differential Equations

Many physical phenomena can be modeled using the language of calculus. For example, observational evidence suggests that the temperature of a cup of tea (or some other liquid) in a room of constant temperature will cool over time at a rate proportional to the difference between the room temperature and the temperature of the tea.

In symbols, if t is the time, M is the room temperature, and $f(t)$ is the temperature of the tea at time t then $f'(t) = k(M - f(t))$ where $k > 0$ is a constant which will depend on the kind of tea (or more generally the kind of liquid) but not on the room temperature or the temperature of the tea. This is **Newton's law of cooling** and the equation that we just wrote down is an example of a **differential equation**. Ideally we would like to solve this equation, namely, find the function $f(t)$ that describes the temperature over time, though this often turns out to be impossible, in which case various approximation techniques must be used. The use and solution of differential equations is an important field of mathematics; here we see how to solve some simple but useful types of differential equation.

Informally, a differential equation is an equation in which one or more of the derivatives of some function appears. Typically, a scientific theory will produce a differential equation (or a system of differential equations) that describes or governs some physical process, but the theory will not produce the desired function or functions directly.

5.1 First Order Differential Equations

We start by considering equations in which only the first derivative of the function appears.

Definition 5.102: First Order Differential Equation

A **first order differential equation** *is an equation of the form* $F(t, y, y') = 0$*. A solution of a first order differential equation is a function* $f(t)$ *that makes* $F(t, f(t), f'(t)) = 0$ *for every value of* t.

Here, F is a function of three variables which we label t, y, and y'. It is understood that y' will explicitly appear in the equation although t and y need not. The term "first order" means that the first derivative of y appears, but no higher order derivatives do.

Example 5.103: Newton's Law of Cooling

The equation from Newton's law of cooling, $y' = k(M - y)$ *is a first order differential equation;* $F(t, y, y') = k(M - y) - y'$.

Example 5.104: A First Order Differential Equation

$y' = t^2 + 1$ *is a first order differential equation;* $F(t, y, y') = y' - t^2 - 1$*. All solutions to this equation are of the form* $t^3/3 + t + C$.

Definition 5.105: First Order Initial Value Problem

A **first order initial value problem** *is a system of equations of the form* $F(t,y,y') = 0$, $y(t_0) = y_0$. *Here* t_0 *is a fixed time and* y_0 *is a number. A solution of an initial value problem is a solution* $f(t)$ *of the differential equation that also satisfies the* **initial condition** $f(t_0) = y_0$.

Example 5.106: An Initial Value Problem

Verify that the initial value problem $y' = t^2 + 1$, $y(1) = 4$ *has solution* $f(t) = t^3/3 + t + 8/3$.

Solution. Observe that $f'(t) = t^2 + 1$ and $f(1) = 1^3/2 + 1 + 8/3 = 4$ as required. ♣

The general first order equation is too general, so we can't describe methods that will work on them all, or even a large portion of them. We can make progress with specific kinds of first order differential equations. For example, much can be said about equations of the form $y' = \phi(t,y)$ where ϕ is a function of the two variables t and y. Under reasonable conditions on ϕ, such an equation has a solution and the corresponding initial value problem has a unique solution. However, in general, these equations can be very difficult or impossible to solve explicitly.

Example 5.107: IVP for Newton's Law of Cooling

Consider this specific example of an initial value problem for Newton's law of cooling: $y' = 2(25 - y)$, $y(0) = 40$. *Discuss the solutions for this initial value problem.*

Solution. We first note the zero of the equation: If $y(t_0) = 25$, the right hand side of the differential equation is zero, and so the constant function $y(t) = 25$ is a solution to the differential equation. It is not a solution to the initial value problem, since $y(0) \neq 25$. (The physical interpretation of this constant solution is that if a liquid is at the same temperature as its surroundings, then the liquid will stay at that temperature.) So long as y is not 25, we can rewrite the differential equation as

$$\begin{aligned} \frac{dy}{dt}\frac{1}{25-y} &= 2 \\ \frac{1}{25-y}\,dy &= 2\,dt, \end{aligned}$$

so

$$\int \frac{1}{25-y}\,dy = \int 2\,dt,$$

that is, the two anti-derivatives must be the same except for a constant difference. We can calculate these anti-derivatives and rearrange the results:

$$\begin{aligned} \int \frac{1}{25-y}\,dy &= \int 2\,dt \\ (-1)\ln|25-y| &= 2t + C_0 \\ \ln|25-y| &= -2t - C_0 = -2t + C \\ |25-y| &= e^{-2t+C} = e^{-2t}e^{C} \\ y - 25 &= \pm e^{C}e^{-2t} \\ y &= 25 \pm e^{C}e^{-2t} = 25 + Ae^{-2t}. \end{aligned}$$

Here $A = \pm e^{C} = \pm e^{-C_0}$ is some non-zero constant. Since we want $y(0) = 40$, we substitute and solve for A:

$$40 = 25 + Ae^{0}$$

$$15 = A,$$

and so $y = 25 + 15e^{-2t}$ is a solution to the initial value problem. Note that y is never 25, so this makes sense for all values of t. However, if we allow $A = 0$ we get the solution $y = 25$ to the differential equation, which would be the solution to the initial value problem if we were to require $y(0) = 25$. Thus, $y = 25 + Ae^{-2t}$ describes all solutions to the differential equation $y' = 2(25 - y)$, and all solutions to the associated initial value problems. ♣

Why could we solve this problem? Our solution depended on rewriting the equation so that all instances of y were on one side of the equation and all instances of t were on the other. Of course, in this case the only t was originally hidden, since we didn't write dy/dt in the original equation. This is not required, however.

Example 5.108: Solving an IVP

Solve the differential equation $y' = 2t(25 - y)$.

Solution. This is almost identical to the previous example. As before, $y(t) = 25$ is a solution. If $y \neq 25$,

$$\begin{aligned}
\int \frac{1}{25-y}\,dy &= \int 2t\,dt \\
(-1)\ln|25-y| &= t^2 + C_0 \\
\ln|25-y| &= -t^2 - C_0 = -t^2 + C \\
|25-y| &= e^{-t^2+C} = e^{-t^2}e^C \\
y - 25 &= \pm e^C e^{-t^2} \\
y &= 25 \pm e^C e^{-t^2} = 25 + Ae^{-t^2}.
\end{aligned}$$

As before, all solutions are represented by $y = 25 + Ae^{-t^2}$, allowing A to be zero. ♣

Definition 5.109: Separable Differential Equations

A first order differential equation is **separable** *if it can be written in the form*

$$y' = f(t)g(y).$$

As in the examples, we can attempt to solve a separable equation by converting to the form

$$\int \frac{1}{g(y)}\,dy = \int f(t)\,dt.$$

This technique is called **separation of variables**. The simplest (in principle) sort of separable equation is one in which $g(y) = 1$, in which case we attempt to solve

$$\int 1\,dy = \int f(t)\,dt.$$

We can do this if we can find an anti-derivative of $f(t)$.

As we have seen so far, a differential equation typically has an infinite number of solutions. Such a solution is called a **general solution**. A corresponding initial value problem will give rise to just one solution. Such a solution in which there are no unknown constants remaining is called a **particular solution**.

The general approach to separable equations is as follows: Suppose we wish to solve $y' = f(t)g(y)$ where f and g are continuous functions. If $g(a) = 0$ for some a then $y(t) = a$ is a constant solution of the equation, since in this case $y' = 0 = f(t)g(a)$. For example, $y' = y^2 - 1$ has constant solutions $y(t) = 1$ and $y(t) = -1$.

To find the nonconstant solutions, we note that the function $1/g(y)$ is continuous where $g \neq 0$, so $1/g$ has an antiderivative G. Let F be an antiderivative of f. Now we write

$$G(y) = \int \frac{1}{g(y)}\,dy = \int f(t)\,dt = F(t) + C,$$

so $G(y) = F(t) + C$. Now we solve this equation for y.

Of course, there are a few places this ideal description could go wrong: We need to be able to find the antiderivatives G and F, and we need to solve the final equation for y. The upshot is that the solutions to the original differential equation are the constant solutions, if any, and all functions y that satisfy $G(y) = F(t) + C$.

Example 5.110: Population Growth and Radioactive Decay

Analyze the differential equation $y' = ky$.

Solution. When $k > 0$, this describes certain simple cases of population growth: It says that the change in the population y is proportional to the population. The underlying assumption is that each organism in the current population reproduces at a fixed rate, so the larger the population the more new organisms are produced. While this is too simple to model most real populations, it is useful in some cases over a limited time. When $k < 0$, the differential equation describes a quantity that decreases in proportion to the current value; this can be used to model radioactive decay.

The constant solution is $y(t) = 0$; of course this will not be the solution to any interesting initial value problem. For the non-constant solutions, we proceed much as before:

$$\begin{aligned}
\int \frac{1}{y}\,dy &= \int k\,dt \\
\ln|y| &= kt + C \\
|y| &= e^{kt}e^{C} \\
y &= \pm e^{C}e^{kt} \\
y &= Ae^{kt}.
\end{aligned}$$

Again, if we allow $A = 0$ this includes the constant solution, and we can simply say that $y = Ae^{kt}$ is the general solution. With an initial value we can easily solve for A to get the solution of the initial value problem. In particular, if the initial value is given for time $t = 0$, $y(0) = y_0$, then $A = y_0$ and the solution is $y = y_0e^{kt}$. ♣

Exercises for 5.1

Exercise 5.1.1 *Which of the following equations are separable?*

(a) $y' = \sin(ty)$

(b) $y' = e^t e^y$

(c) $yy' = t$

(d) $y' = (t^3 - t)\arcsin(y)$

(e) $y' = t^2 \ln y + 4t^3 \ln y$

Exercise 5.1.2 *Solve* $y' = 1/(1+t^2)$.

Exercise 5.1.3 *Solve the initial value problem* $y' = t^n$ *with* $y(0) = 1$ *and* $n \geq 0$.

Exercise 5.1.4 *Solve* $y' = \ln t$.

Exercise 5.1.5 *Identify the constant solutions (if any) of* $y' = t \sin y$.

Exercise 5.1.6 *Identify the constant solutions (if any) of* $y' = te^y$.

Exercise 5.1.7 *Solve* $y' = t/y$.

Exercise 5.1.8 *Solve* $y' = y^2 - 1$.

Exercise 5.1.9 *Solve* $y' = t/(y^3 - 5)$. *You may leave your solution in implicit form: that is, you may stop once you have done the integration, without solving for y.*

Exercise 5.1.10 *Find a non-constant solution of the initial value problem* $y' = y^{1/3}$, $y(0) = 0$, *using separation of variables. Note that the constant function* $y(t) = 0$ *also solves the initial value problem. This shows that an initial value problem can have more than one solution.*

Exercise 5.1.11 *Solve the equation for Newton's law of cooling leaving M and k unknown.*

Exercise 5.1.12 *After 10 minutes in Jean-Luc's room, his tea has cooled to* $40°$ *Celsius from* $100°$ *Celsius. The room temperature is* $25°$ *Celsius. How much longer will it take to cool to* $35°$*?*

Exercise 5.1.13 *Solve the* **logistic equation** $y' = ky(M - y)$. *(This is a somewhat more reasonable population model in most cases than the simpler* $y' = ky$*.) Sketch the graph of the solution to this equation when* $M = 1000$, $k = 0.002$, $y(0) = 1$.

Exercise 5.1.14 *Suppose that* $y' = ky$, $y(0) = 2$, *and* $y'(0) = 3$. *What is y?*

Exercise 5.1.15 *A radioactive substance obeys the equation* $y' = ky$ *where* $k < 0$ *and y is the mass of the substance at time t. Suppose that initially, the mass of the substance is* $y(0) = M > 0$. *At what time does half of the mass remain? (This is known as the half life. Note that the half life depends on k but not on M.)*

Exercise 5.1.16 *Bismuth-210 has a half life of five days. If there is initially 600 milligrams, how much is left after 6 days? When will there be only 2 milligrams left?*

Exercise 5.1.17 *The half life of carbon-14 is 5730 years. If one starts with 100 milligrams of carbon-14, how much is left after 6000 years? How long do we have to wait before there is less than 2 milligrams?*

Exercise 5.1.18 *A certain species of bacteria doubles its population (or its mass) every hour in the lab. The differential equation that models this phenomenon is* $y' = ky$, *where* $k > 0$ *and y is the population of bacteria at time t. What is y?*

Exercise 5.1.19 *If a certain microbe doubles its population every 4 hours and after 5 hours the total population has mass 500 grams, what was the initial mass?*

5.2 First Order Homogeneous Linear Equations

A simple, but important and useful, type of separable equation is the **first order homogeneous linear equation**:

Definition 5.111: First Order Homogeneous Linear Equation

A first order homogeneous linear differential equation is one of the form $y' + p(t)y = 0$ *or equivalently* $y' = -p(t)y$.

"Homogeneous" refers to the zero on the right side of the equation, provided that y' and y are on the left. "Linear" in this definition indicates that both y' and y appear independently and explicitly; we don't see y' or y to any power greater than 1, or multiplied by each other (i.e. $y'y$).

Example 5.112: Linear Examples

The equation $y' = 2t(25-y)$ *can be written* $y' + 2ty = 50t$. *This is linear, but not homogeneous. The equation* $y' = ky$, *or* $y' - ky = 0$ *is linear and homogeneous, with a particularly simple* $p(t) = -k$. *The equation* $y' + y^2 = 0$ *is homogeneous, but not linear.*

Since first order homogeneous linear equations are separable, we can solve them in the usual way:

$$\begin{aligned} y' &= -p(t)y \\ \int \frac{1}{y}\,dy &= \int -p(t)\,dt \\ \ln|y| &= P(t) + C \\ y &= \pm e^{P(t)} \\ y &= Ae^{P(t)}, \end{aligned}$$

where $P(t)$ is an anti-derivative of $-p(t)$. As in previous examples, if we allow $A = 0$ we get the constant solution $y = 0$.

Example 5.113: Solving an IVP

Solve the initial value problem

$$y' + y\cos t = 0,$$

subject to $y(0) = 1/2$ *and* $y(2) = 1/2$.

Solution. We start with

$$P(t) = \int -\cos t\,dt = -\sin t,$$

so the general solution to the differential equation is

$$y = Ae^{-\sin t}.$$

To compute A we substitute:

$$\frac{1}{2} = Ae^{-\sin 0} = A,$$

so the solutions is

$$y = \frac{1}{2}e^{-\sin t}.$$

For the second problem,

$$\begin{aligned}\frac{1}{2} &= Ae^{-\sin 2} \\ A &= \frac{1}{2}e^{\sin 2}\end{aligned}$$

so the solution is

$$y = \frac{1}{2}e^{\sin 2}e^{-\sin t}.$$

♣

Example 5.114:

Solve the initial value problem $ty' + 3y = 0$, $y(1) = 2$, *assuming* $t > 0$.

Solution. We write the equation in standard form: $y' + 3y/t = 0$. Then

$$P(t) = \int -\frac{3}{t}\,dt = -3\ln t$$

and

$$y = Ae^{-3\ln t} = At^{-3}.$$

Substituting to find A: $2 = A(1)^{-3} = A$, so the solution is $y = 2t^{-3}$. ♣

Exercises for 5.2

Find the general solution of each equation in the following exercises.

Exercise 5.2.1 $y' + 5y = 0$

Exercise 5.2.2 $y' - 2y = 0$

Exercise 5.2.3 $y' + \dfrac{y}{1+t^2} = 0$

Exercise 5.2.4 $y' + t^2y = 0$

In the following exercises, solve the initial value problem.

Exercise 5.2.5 $y' + y = 0$, $y(0) = 4$

Exercise 5.2.6 $y' - 3y = 0$, $y(1) = -2$

Exercise 5.2.7 $y' + y\sin t = 0$, $y(\pi) = 1$

Exercise 5.2.8 $y' + ye^t = 0$, $y(0) = e$

Exercise 5.2.9 $y' + y\sqrt{1+t^4} = 0$, $y(0) = 0$

Exercise 5.2.10 $y' + y\cos(e^t) = 0$, $y(0) = 0$

Exercise 5.2.11 $ty' - 2y = 0$, $y(1) = 4$

Exercise 5.2.12 $t^2y' + y = 0$, $y(1) = -2$, $t > 0$

Exercise 5.2.13 $t^3y' = 2y$, $y(1) = 1$, $t > 0$

Exercise 5.2.14 $t^3y' = 2y$, $y(1) = 0$, $t > 0$

Exercise 5.2.15 *A function $y(t)$ is a solution of $y' + ky = 0$. Suppose that $y(0) = 100$ and $y(2) = 4$. Find k and find $y(t)$.*

Exercise 5.2.16 *A function $y(t)$ is a solution of $y' + t^k y = 0$. Suppose that $y(0) = 1$ and $y(1) = e^{-13}$. Find k and find $y(t)$.*

Exercise 5.2.17 *A bacterial culture grows at a rate proportional to its population. If the population is one million at $t = 0$ and 1.5 million at $t = 1$ hour, find the population as a function of time.*

Exercise 5.2.18 *A radioactive element decays with a half-life of 6 years. If a mass of the element weighs ten pounds at $t = 0$, find the amount of the element at time t.*

5.3 First Order Linear Equations

As you might guess, a first order linear differential equation has the form $y' + p(t)y = f(t)$. Not only is this closely related in form to the first order homogeneous linear equation, we can use what we know about solving homogeneous equations to solve the general linear equation.

Suppose that $y_1(t)$ and $y_2(t)$ are solutions to $y' + p(t)y = f(t)$. Let $g(t) = y_1 - y_2$. Then

$$\begin{aligned} g'(t) + p(t)g(t) &= y_1' - y_2' + p(t)(y_1 - y_2) \\ &= (y_1' + p(t)y_1) - (y_2' + p(t)y_2) \\ &= f(t) - f(t) = 0. \end{aligned}$$

In other words, $g(t) = y_1 - y_2$ is a solution to the homogeneous equation $y' + p(t)y = 0$. Turning this around, any solution to the linear equation $y' + p(t)y = f(t)$, call it y_1, can be written as $y_2 + g(t)$, for some particular y_2 and some solution $g(t)$ of the homogeneous equation $y' + p(t)y = 0$. Since we already know how to find all solutions of the homogeneous equation, finding just one solution to the equation $y' + p(t)y = f(t)$ will give us all of them.

How might we find that one particular solution to $y' + p(t)y = f(t)$? Again, it turns out that what we already know helps. We know that the general solution to the homogeneous equation $y' + p(t)y = 0$ looks like $Ae^{P(t)}$. We now make an inspired guess: Consider the function $v(t)e^{P(t)}$, in which we have replaced the constant parameter A with the function $v(t)$. This technique is called **variation of parameters**. For convenience write this as $s(t) = v(t)h(t)$, where $h(t) = e^{P(t)}$ is a solution to the homogeneous equation. Now let's compute a bit with $s(t)$:

$$\begin{aligned} s'(t) + p(t)s(t) &= v(t)h'(t) + v'(t)h(t) + p(t)v(t)h(t) \\ &= v(t)(h'(t) + p(t)h(t)) + v'(t)h(t) \\ &= v'(t)h(t). \end{aligned}$$

The last equality is true because $h'(t) + p(t)h(t) = 0$. Since $h(t)$ is a solution to the homogeneous equation. We are hoping to find a function $s(t)$ so that $s'(t) + p(t)s(t) = f(t)$; we will have such a function if we can arrange to have $v'(t)h(t) = f(t)$, that is, $v'(t) = f(t)/h(t)$. But this is as easy (or hard) as finding an anti-derivative of $f(t)/h(t)$. Putting this all together, the general solution to $y' + p(t)y = f(t)$ is

$$v(t)h(t) + Ae^{P(t)} = v(t)e^{P(t)} + Ae^{P(t)}.$$

Example 5.115: Solving an IVP

Find the solution of the initial value problem $y' + 3y/t = t^2$, $y(1) = 1/2$.

Solution. First we find the general solution; since we are interested in a solution with a given condition at $t = 1$, we may assume $t > 0$. We start by solving the homogeneous equation as usual; call the solution g:

$$g = Ae^{-\int(3/t)\,dt} = Ae^{-3\ln t} = At^{-3}.$$

Then as in the discussion, $h(t) = t^{-3}$ and $v'(t) = t^2/t^{-3} = t^5$, so $v(t) = t^6/6$. We know that every solution to the equation looks like

$$v(t)t^{-3} + At^{-3} = \frac{t^6}{6}t^{-3} + At^{-3} = \frac{t^3}{6} + At^{-3}.$$

Finally we substitute to find A:

$$\begin{aligned} \frac{1}{2} &= \frac{(1)^3}{6} + A(1)^{-3} = \frac{1}{6} + A \\ A &= \frac{1}{2} - \frac{1}{6} = \frac{1}{3}. \end{aligned}$$

The solution is then

$$y = \frac{t^3}{6} + \frac{1}{3}t^{-3}.$$

♣

Another common method for solving such a differential equation is by means of an **integrating factor**. In the differential equation $y' + p(t)y = f(t)$, we note that if we multiply through by a function $I(t)$ to get $I(t)y' + I(t)p(t)y = I(t)f(t)$, the left hand side looks like it could be a derivative computed by the product rule:

$$\frac{d}{dt}(I(t)y) = I(t)y' + I'(t)y.$$

Now if we could choose $I(t)$ so that $I'(t) = I(t)p(t)$, this would be exactly the left hand side of the differential equation. But this is just a first order homogeneous linear equation, and we know a solution is $I(t) = e^{Q(t)}$, where $Q(t) = \int p\,dt$; note that $Q(t) = -P(t)$, where $P(t)$ appears in the variation of parameters method and $P'(t) = -p$. Now the modified differential equation is

$$\begin{aligned} e^{-P(t)}y' + e^{-P(t)}p(t)y &= e^{-P(t)}f(t) \\ \frac{d}{dt}(e^{-P(t)}y) &= e^{-P(t)}f(t). \end{aligned}$$

Integrating both sides gives

$$\begin{aligned} e^{-P(t)}y &= \int e^{-P(t)}f(t)\,dt \\ y &= e^{P(t)}\int e^{-P(t)}f(t)\,dt. \end{aligned}$$

If you look carefully, you will see that this is exactly the same solution we found by variation of parameters, because $e^{-P(t)}f(t) = f(t)/h(t)$.

Some people find it easier to remember how to use the integrating factor method, rather than variation of parameters. Since ultimately they require the same calculation, you should use whichever of the two methods appeals to you more strongly. Using this method, the solution of the previous example would look just a bit different: Starting with $y' + 3y/t = t^2$, we recall that the integrating factor is $e^{\int 3/t} = e^{3\ln t} = t^3$. Then we multiply through by the integrating factor and solve:

$$\begin{aligned} t^3y' + t^3 3y/t &= t^3t^2 \\ t^3y' + t^2 3y &= t^5 \\ \frac{d}{dt}(t^3y) &= t^5 \\ t^3y &= t^6/6 \\ y &= t^3/6. \end{aligned}$$

This is the same answer, of course, and the problem is then finished just as before.

Exercises for 5.3

In the following exercises, find the general solution of the equation.

Exercise 5.3.1 $y' + 4y = 8$

Exercise 5.3.2 $y' - 2y = 6$

Exercise 5.3.3 $y' + ty = 5t$

Exercise 5.3.4 $y' + e^t y = -2e^t$

Exercise 5.3.5 $y' - y = t^2$

Exercise 5.3.6 $2y' + y = t$

Exercise 5.3.7 $ty' - 2y = 1/t$, $t > 0$

Exercise 5.3.8 $ty' + y = \sqrt{t}$, $t > 0$

Exercise 5.3.9 $y' \cos t + y \sin t = 1$, $-\pi/2 < t < \pi/2$

Exercise 5.3.10 $y' + y \sec t = \tan t$, $-\pi/2 < t < \pi/2$

5.4 Approximation

We have seen how to solve a restricted collection of differential equations, or more accurately, how to attempt to solve them—we still may not be able to find the required anti-derivatives. Not surprisingly, non-linear equations can be even more difficult to solve. Yet much is known about solutions to some more general equations.

Suppose $\phi(t,y)$ is a function of two variables. A more general class of first order differential equations has the form $y' = \phi(t,y)$. This is not necessarily a linear first order equation, since ϕ may depend on y in some complicated way; note however that y' appears in a very simple form. Under suitable conditions on the function ϕ, it can be shown that every such differential equation has a solution, and moreover that for each initial condition the associated initial value problem has exactly one solution. In practical applications this is obviously a very desirable property.

Example 5.116: First Order Non-linear

The equation $y' = t - y^2$ is a first order non-linear equation, because y appears to the second power. We will not be able to solve this equation.

Example 5.117: Non-linear and Separable

The equation $y' = y^2$ is also non-linear, but it is separable and can be solved by separation of variables.

Not all differential equations that are important in practice can be solved exactly, so techniques have been developed to approximate solutions. We describe one such technique, **Euler's Method**, which is simple though not particularly useful compared to some more sophisticated techniques.

Suppose we wish to approximate a solution to the initial value problem $y' = \phi(t,y)$, $y(t_0) = y_0$, for $t \geq t_0$. Under reasonable conditions on ϕ, we know the solution exists, represented by a curve in the t-y plane; call this solution $f(t)$. The point (t_0, y_0) is of course on this curve. We also know the slope of the curve at this point, namely $\phi(t_0, y_0)$. If we follow the tangent line for a brief distance, we arrive at a point that should be almost on the graph of $f(t)$, namely $(t_0 + \Delta t, y_0 + \phi(t_0, y_0)\Delta t)$; call this point (t_1, y_1). Now we pretend, in effect, that this point really is on the graph of $f(t)$, in which case we again know the slope of the curve through (t_1, y_1), namely $\phi(t_1, y_1)$. So we can compute a new point, $(t_2, y_2) = (t_1 + \Delta t, y_1 + \phi(t_1, y_1)\Delta t)$ that is a little farther along, still close to the graph of $f(t)$ but probably not quite so close as (t_1, y_1). We can continue in this way, doing a sequence of straightforward calculations, until we have an approximation (t_n, y_n) for whatever time t_n we need. At each step we do essentially the same calculation, namely:

$$(t_{i+1}, y_{i+1}) = (t_i + \Delta t, y_i + \phi(t_i, y_i)\Delta t).$$

We expect that smaller time steps Δt will give better approximations, but of course it will require more work to compute to a specified time. It is possible to compute a guaranteed upper bound on how far off the approximation might be, that is, how far y_n is from $f(t_n)$. Suffice it to say that the bound is not particularly good and that there are other more complicated approximation techniques that do better.

Example 5.118: Approximating a Solution

Compute an approximation to the solution for $y' = t - y^2$, $y(0) = 0$, when $t = 1$.

Solution. We will use $\Delta t = 0.2$, which is easy to do even by hand, though we should not expect the resulting approximation to be very good. We get

$$\begin{aligned}
(t_1, y_1) &= (0 + 0.2, 0 + (0 - 0^2)0.2) = (0.2, 0) \\
(t_2, y_2) &= (0.2 + 0.2, 0 + (0.2 - 0^2)0.2) = (0.4, 0.04) \\
(t_3, y_3) &= (0.6, 0.04 + (0.4 - 0.04^2)0.2) = (0.6, 0.11968) \\
(t_4, y_4) &= (0.8, 0.11968 + (0.6 - 0.11968^2)0.2) = (0.8, 0.23681533952) \\
(t_5, y_5) &= (1.0, 0.23681533952 + (0.6 - 0.23681533952^2)0.2) = (1.0, 0.385599038513605)
\end{aligned}$$

So $y(1) \approx 0.3856$. As it turns out, this is not accurate to even one decimal place. Figure 5.35 shows these points connected by line segments (the lower curve) compared to a solution obtained by a much better approximation technique. Note that the shape is approximately correct even though the end points are quite far apart.

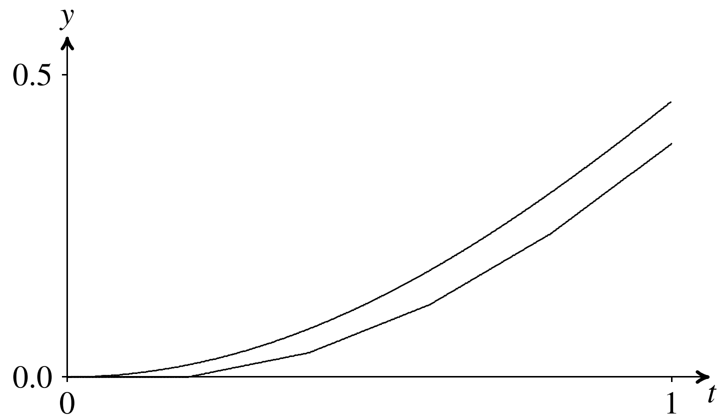


Figure 5.35: Approximating a solution to $y' = t - y^2$**,** $y(0) = 0$**.**

If you need to do Euler's method by hand, it is useful to construct a table to keep track of the work, as shown in Figure 5.36. Each row holds the computation for a single step: The starting point (t_i, y_i); the stepsize Δt; the computed slope $\phi(t_i, y_i)$; the change in y, $\Delta y = \phi(t_i, y_i)\Delta t$; and the new point, $(t_{i+1}, y_{i+1}) = (t_i + \Delta t, y_i + \Delta y)$. The starting point in each row is the newly computed point from the end of the previous row.

(t,y)	Δt	$\phi(t,y)$	$\Delta y = \phi(t,y)\Delta t$	$(t+\Delta t, y+\Delta y)$
$(0,0)$	0.2	0	0	$(0.2,0)$
$(0.2,0)$	0.2	0.2	0.04	$(0.4,0.04)$
$(0.4,0.04)$	0.2	0.3984	0.07968	$(0.6,0.11968)$
$(0.6,0.11968)$	0.2	$0.58\ldots$	$0.117\ldots$	$(0.8,0.236\ldots)$
$(0.8,0.236\ldots)$	0.2	$0.743\ldots$	$0.148\ldots$	$(1.0,0.385\ldots)$

Figure 5.36: Computing with Euler's Method.

♣

Euler's method is related to another technique that can help in understanding a differential equation in a qualitative way. Euler's method is based on the ability to compute the slope of a solution curve at any point in the plane, simply by computing $\phi(t,y)$. If we compute $\phi(t,y)$ at many points, say in a grid, and plot a small line segment with that slope at the point, we can get an idea of how solution curves must look. Such a plot is called a **slope field**. A slope field for $\phi = t - y^2$ is shown in Figure 5.37; compare this to figure 5.35. With a little practice, one can sketch reasonably accurate solution curves based on the slope field, in essence doing Euler's method visually.

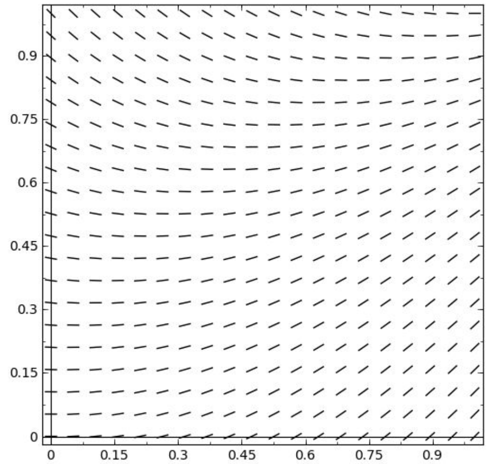
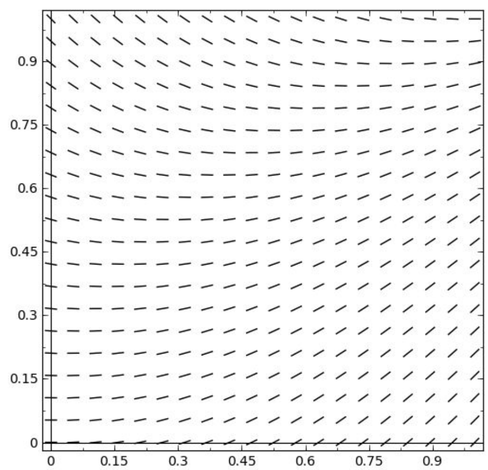


Figure 5.37: A slope field for $y' = t - y^2$.



Even when a differential equation can be solved explicitly, the slope field can help in understanding what the solutions look like with various initial conditions. Recall the logistic equation $y' = ky(M - y)$: y is a population at time t, M is a measure of how large a population the environment can support, and k measures the reproduction rate of the population. Figure 5.38 shows a slope field for this equation that is quite informative. It is apparent that if the initial population is smaller than M it rises to M over the long term, while if the initial population is greater than M it decreases to M.

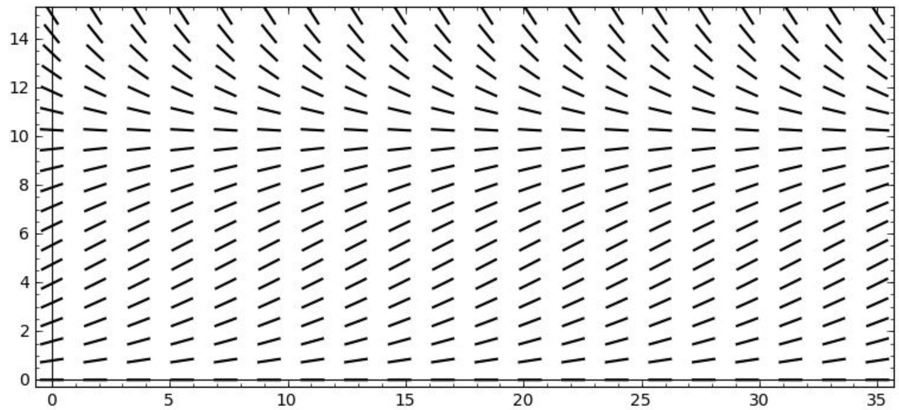


Figure 5.38: A slope field for $y' = 0.2y(10-y)$.

Exercises for 5.4

In the following exercises, compute the Euler approximations for the initial value problem for $0 \le t \le 1$ and $\Delta t = 0.2$. If you have access to Sage, generate the slope field first and attempt to sketch the solution curve. Then use Sage to compute better approximations with smaller values of Δt.

Exercise 5.4.1 $y' = t/y$, $y(0) = 1$

Exercise 5.4.2 $y' = t + y^3$, $y(0) = 1$

Exercise 5.4.3 $y' = \cos(t+y)$, $y(0) = 1$

Exercise 5.4.4 $y' = t \ln y$, $y(0) = 2$

Unit 6: Sequences and Infinite Series

Consider the following sum:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{2^i}+\cdots$$

The dots at the end indicate that the sum goes on forever. Does this make sense? Can we assign a numerical value to an infinite sum? While at first it may seem difficult or impossible, we have certainly done something similar when we talked about one quantity getting "closer and closer" to a fixed quantity. Here we could ask whether, as we add more and more terms, the sum gets closer and closer to some fixed value. That is, look at

$$\begin{aligned}\frac{1}{2}&=\frac{1}{2}\\ \frac{3}{4}&=\frac{1}{2}+\frac{1}{4}\\ \frac{7}{8}&=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\\ \frac{15}{16}&=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}\end{aligned}$$

and so on, and consider whether these values have a limit. It seems likely that they do, namely 1. In fact, as we will see, it's not hard to show that

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{2^i}=\frac{2^i-1}{2^i}=1-\frac{1}{2^i}$$

and then

$$\lim_{i\to\infty}1-\frac{1}{2^i},$$

which gets closer and closer to 1 as i gets larger.

There is a context in which we already implicitly accept this notion of infinite sum without really thinking of it as a sum: The representation of a real number as an infinite decimal. For example,

$$0.3333\bar{3}=\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+\frac{3}{10000}+\cdots=\frac{1}{3},$$

or likewise

$$3.14159\ldots=3+\frac{1}{10}+\frac{4}{100}+\frac{1}{1000}+\frac{5}{10000}+\frac{9}{100000}+\cdots=\pi.$$

An infinite sum is called a **series**, and is usually written using sigma notation. In this case, however, we use ∞ to indicate that there is no 'last term'. The series we first examined can be written as

$$\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^i}=\sum_{i=1}^{\infty}\frac{1}{2^i}$$

A related notion that will aid our investigations is that of a **sequence**. A sequence is just an ordered (possibly infinite) list of numbers. For example,

$$1,\frac{1}{2},\frac{1}{3},\ldots,\frac{1}{n},\ldots$$

We will begin by learning some useful facts about sequences.

6.1 Sequences

While the idea of a sequence of numbers, $a_1, a_2, a_3, \ldots$ is straightforward, it is useful to think of a sequence as a function. We have dealt with functions whose domains are the real numbers, or a subset of the real numbers, like $f(x) = \sin x$. A sequence can be regarded as a function with domain as the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ or the non-negative integers, $\mathbb{Z}^{\geq 0} = \{0, 1, 2, 3, \ldots\}$. The range of the function is still allowed to be the set of all real numbers; we say that a sequence is a function $f\colon \mathbb{N} \to \mathbb{R}$. Sequences are commonly denoted in several different, but equally acceptable ways:

$$a_1, a_2, a_3, \ldots$$
$$\{a_n\}_{n=1}^{\infty}$$
$$\{f(n)\}_{n=1}^{\infty}$$

As with functions of the real numbers, we will most often encounter sequences that can be expressed by a formula. We have already seen the sequence $a_i = f(i) = 1 - 1/2^i$. Some other simple examples are:

$$f(i) = \frac{i}{i+1}$$
$$f(n) = \frac{1}{2^n}$$
$$f(n) = \sin(n\pi/6)$$
$$f(i) = \frac{(i-1)(i+2)}{2^i}$$

Frequently these formulas will make sense if thought of either as functions with domain $\mathbb{R}$ or $\mathbb{N}$, though occasionally one will make sense for integer values only.

The main question of interest when dealing with sequences is what happens to the terms as we go further and further down the list. In particular, as i becomes extremely large, does a_i get closer to one specific value? This is reminiscent of a question we asked when looking at limits of functions. In fact, the problems are closely related and we define the limit of a sequence as follows:

Definition 6.119: Limit of a Sequence

Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence. We say that $\lim\limits_{n\to\infty} a_n = L$ if for every $\varepsilon > 0$ there is an $N > 0$ so that whenever $n > N$, $|a_n - L| < \varepsilon$. If $\lim\limits_{n\to\infty} a_n = L$ we say that the sequence **converges** *to L, otherwise it* **diverges**.

Intuitively, $\lim_{n\to\infty} a_2 = L$ means that the further we go in the sequence, the closer the terms get to L.

Example 6.120: Exponential Sequence

Show that $\{2^{1/n}\}_{n=1}^{\infty}$ converges to 1.

Solution. Suppose $\varepsilon > 0$. Then let $N = \frac{1}{\log_2(1+\varepsilon)}$. Note that $N > 0$. Now if $n > N$, then

$$n > \frac{1}{\log_2(1+\varepsilon)}$$

$$\begin{aligned}
\log_2(1+\varepsilon) &> \frac{1}{n} \\
1+\varepsilon &> 2^{1/n} \\
\varepsilon &> 2^{1/n}-1 \\
\varepsilon &> \left|2^{1/n}-1\right|
\end{aligned}$$

♣

Note that we generally need to work "backwards" from the last line of the proof to determine how to choose N. Having done so, we write the actual proof as we have done here to show that this value of N 'works'.

If a sequence is defined by a formula $\{f(i)\}_{i=1}^{\infty}$, we can often expand the domain of the function f to the set of all (or almost all) real numbers. For example, $f(i) = \frac{1}{i}$ is defined for all non-zero real numbers.

When this happens, we can sometimes find the limit of the sequence $\{f(i)\}_{i=1}^{\infty}$ more easily by finding the limit of the function $f(x)$, $x \in \mathbb{R}$, as x approaches infinity.

Theorem 6.121: Limit of a Sequence

If $\lim_{x\to\infty} f(x) = L$, *where* $f : \mathbb{R} \to \mathbb{R}$, *then* $\{f(i)\}_{i=1}^{\infty}$ *converges to* L.

Proof. This follows immediately from earlier definitions and 6.119. ♣

Hereafter we will use the convention that x refers to a real-valued variable and i and n are integer-valued.

Example 6.122: Sequence of $1/n$

Show that $\{\frac{1}{n}\}_{n=0}^{\infty}$ *converges to 0.*

Solution. Since $\lim_{x\to\infty} \frac{1}{x} = 0$, $\lim_{n\to\infty} \frac{1}{n} = 0$. ♣

Note that the converse of Theorem 6.121 is not true.

Let $f(n) = \sin(n\pi)$. This the sequence

$$\sin(0\pi), \sin(1\pi), \sin(2\pi), \sin(3\pi), \ldots = 0, 0, 0, 0, \ldots$$

since $\sin(n\pi) = 0$ when n is an integer. Thus $\lim\limits_{n\to\infty} f(n) = 0$. But $\lim\limits_{x\to\infty} f(x)$, when x is real, does not exist: as x gets bigger and bigger, the values $\sin(x\pi)$ do not get closer and closer to a single value, but take on all values between -1 and 1 over and over. In general, whenever you want to know $\lim\limits_{n\to\infty} f(n)$ you should first attempt to compute $\lim\limits_{x\to\infty} f(x)$, since if the latter exists it is also equal to the first limit. But if for some reason $\lim\limits_{x\to\infty} f(x)$ does not exist, it may still be true that $\lim\limits_{n\to\infty} f(n)$ exists, but you'll have to figure out another way to compute it.

It is occasionally useful to think of the graph of a sequence. Since the function is defined only for integer values, the graph is just a sequence of points. In Figure 6.39 we see the graphs of two sequences and the graphs of the corresponding real functions.

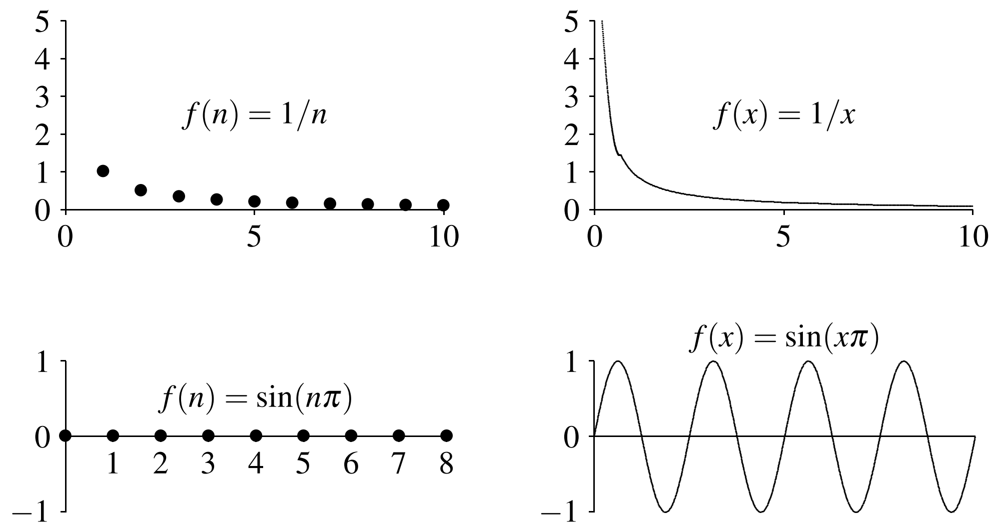


Figure 6.39: Graphs of sequences and their corresponding real functions.

Not surprisingly, the properties of limits of real functions translate into properties of sequences quite easily. The Properties of Limits theorem becomes:

Theorem 6.123: Properties of Sequences

Suppose that $\lim_{n\to\infty} a_n = L$ *and* $\lim_{n\to\infty} b_n = M$ *and* k *is some constant. Then*

$$\lim_{n\to\infty} ka_n = k\lim_{n\to\infty} a_n = kL$$
$$\lim_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = L+M$$
$$\lim_{n\to\infty}(a_n-b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = L-M$$
$$\lim_{n\to\infty}(a_nb_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n = LM$$
$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{L}{M}, \text{ if } M \text{ is not } 0$$

Likewise the Squeeze Theorem becomes:

Theorem 6.124: Squeeze Theorem for Sequences

Suppose that $a_n \leq b_n \leq c_n$ *for all* $n > N$, *for some* N. *If* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, *then* $\lim_{n\to\infty} b_n = L$.

And a final useful fact:

Theorem 6.125: Absolute Value Sequence

$\lim_{n\to\infty} |a_n| = 0$ *if and only if* $\lim_{n\to\infty} a_n = 0$.

This says simply that the size of $|a_n|$ gets close to zero if and only if a_n gets close to zero.

Example 6.126: Convergence of a Rational Fraction

Determine whether $\left\{\frac{n}{n+1}\right\}_{n=0}^{\infty}$ *converges or diverges. If it converges, compute the limit.*

Solution. Defining $f(x) = \frac{x}{x+1}$ we obtain

$$\lim_{x\to\infty} \frac{x}{x+1} = \lim_{x\to\infty} 1 - \frac{1}{x+1} = 1 - 0 = 1.$$

Thus the sequence converges to 1. ♣

Example 6.127: Convergence of Ratio with Natural Logarithm

Determine whether $\left\{\frac{\ln n}{n}\right\}_{n=1}^{\infty}$ *converges or diverges. If it converges, compute the limit.*

Solution. We compute

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0,$$

using L'Hôpital's Rule. Thus the sequence converges to 0. ♣

Example 6.128: Alternating Ones

Determine whether $\{(-1)^n\}_{n=0}^{\infty}$ *converges or diverges. If it converges, compute the limit.*

Solution. $f(x) = (-1)^x$ is undefined for irrational values of x so $\lim_{x\to\infty}(-1)^x$ does not exist. However, the sequence has a very simple pattern:

$$1, -1, 1, -1, 1\dots$$

and clearly diverges. ♣

Example 6.129: Convergence of Exponential

Determine whether $\{(-1/2)^n\}_{n=0}^{\infty}$ *converges or diverges. If it converges, compute the limit.*

Solution. We consider the sequence $\{|(-1/2)^n|\}_{n=0}^{\infty} = \{(1/2)^n\}_{n=0}^{\infty}$. Then

$$\lim_{x\to\infty} \left(\frac{1}{2}\right)^x = \lim_{x\to\infty} \frac{1}{2^x} = 0,$$

so by Theorem 6.125 the sequence converges to 0. ♣

Example 6.130: Using the Squeeze Theorem for Sequences

Determine whether $\{(\sin n)/\sqrt{n}\}_{n=1}^{\infty}$ *converges or diverges. If it converges, compute the limit.*

Solution. Since $|\sin n| \leq 1$, $0 \leq |\sin n/\sqrt{n}| \leq 1/\sqrt{n}$ and we can use Theorem 6.124 with $a_n = 0$ and $c_n = 1/\sqrt{n}$. Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = 0$, $\lim_{n\to\infty} \sin n/\sqrt{n} = 0$ and the sequence converges to 0. ♣

Example 6.131: Geometric Sequence

Let r be a fixed real number. Determine when $\{r^n\}_{n=0}^{\infty}$ coverges.

Solution. A particularly common and useful sequence is $\{r^n\}_{n=0}^{\infty}$, for various values of r. Some are quite easy to understand: If $r = 1$ the sequence converges to 1 since every term is 1, and likewise if $r = 0$ the sequence converges to 0. If $r = -1$ this is the sequence of Example 6.128 and diverges. If $r > 1$ or $r < -1$ the terms r^n get large without limit, so the sequence diverges. If $0 < r < 1$ then the sequence converges to 0. If $-1 < r < 0$ then $|r^n| = |r|^n$ and $0 < |r| < 1$, so the sequence $\{|r|^n\}_{n=0}^{\infty}$ converges to 0, so also $\{r^n\}_{n=0}^{\infty}$ converges to 0. converges. In summary, $\{r^n\}$ converges precisely when $-1 < r \leq 1$ in which case

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

♣

Sequences of this form, or the more general form $\{kr^n\}_{n=0}^{\infty}$, are called **geometric sequences** or **geometric progressions**. They are encountered in a large variety of mathematical and real-world applications.

Sometimes we will not be able to determine the limit of a sequence, but we still would like to know whether it converges. In some cases we can determine this even without being able to compute the limit.

A sequence is called **increasing** or sometimes **strictly increasing** if $a_i < a_{i+1}$ for all i. It is called **non-decreasing** or sometimes (unfortunately) **increasing** if $a_i \leq a_{i+1}$ for all i. Similarly a sequence is **decreasing** if $a_i > a_{i+1}$ for all i and **non-increasing** if $a_i \geq a_{i+1}$ for all i. If a sequence has any of these properties it is called **monotonic**.

Example 6.132:

The sequence

$$\left\{\frac{2^i - 1}{2^i}\right\}_{i=1}^{\infty} = \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots,$$

is increasing, and

$$\left\{\frac{n+1}{n}\right\}_{i=1}^{\infty} = \frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots$$

is decreasing.

A sequence is **bounded above** if there is some number N such that $a_n \leq N$ for every n, and **bounded below** if there is some number N such that $a_n \geq N$ for every n. If a sequence is bounded above and bounded below it is **bounded**. If a sequence $\{a_n\}_{n=0}^{\infty}$ is increasing or non-decreasing it is bounded below (by a_0), and if it is decreasing or non-increasing it is bounded above (by a_0). Finally, with all this new terminology we can state an important theorem.

Theorem 6.133: Bounded Monotonic Sequence

If a sequence is bounded and monotonic then it converges.

We will not prove this, but the proof appears in many calculus books. It is not hard to believe: suppose that a sequence is increasing and bounded, so each term is larger than the one before, yet never larger than some fixed value N. The terms must then get closer and closer to some value between a_0 and N. It need not be N, since N may be a "too-generous" upper bound; the limit will be the smallest number that is above all of the terms a_i.

Example 6.134:

Determine whether $\left\{\dfrac{2^i-1}{2^i}\right\}_{i=1}^{\infty}$ *converges.*

Solution. For every $i \geq 1$ we have $0 < (2^i-1)/2^i < 1$, so the sequence is bounded, and we have already observed that it is necessary. Therefore, the sequence converges. ♣

We don't actually need to know that a sequence is monotonic to apply this theorem—it is enough to know that the sequence is "eventually" monotonic, that is, that at some point it becomes increasing or decreasing. For example, the sequence 10, 9, 8, 15, 3, 21, 4, $3/4$, $7/8$, $15/16$, $31/32, \ldots$ is not increasing, because among the first few terms it is not. But starting with the term $3/4$ it is increasing, so the theorem tells us that the sequence $3/4, 7/8, 15/16, 31/32, \ldots$ converges. Since convergence depends only on what happens as n gets large, adding a few terms at the beginning can't turn a convergent sequence into a divergent one.

Example 6.135:

Show that $\{n^{1/n}\}$ *converges.*

Solution. We first show that this sequence is decreasing, that is, that $n^{1/n} > (n+1)^{1/(n+1)}$. Consider the real function $f(x) = x^{1/x}$ when $x \geq 1$. We can compute the derivative, $f'(x) = x^{1/x}(1-\ln x)/x^2$, and note that when $x \geq 3$ this is negative. Since the function has negative slope, $n^{1/n} > (n+1)^{1/(n+1)}$ when $n \geq 3$. Since all terms of the sequence are positive, the sequence is decreasing and bounded when $n \geq 3$, and so the sequence converges. (As it happens, we can compute the limit in this case, but we know it converges even without knowing the limit; see Exercise 6.1.1.) ♣

Example 6.136:

Show that $\{n!/n^n\}$ *converges.*

Solution. If we look at the ratio of successive terms we see that:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}}\frac{n^n}{n!} = \frac{(n+1)!}{n!}\frac{n^n}{(n+1)^{n+1}} = \frac{n+1}{n+1}\left(\frac{n}{n+1}\right)^n = \left(\frac{n}{n+1}\right)^n < 1.$$

Therefore $a_{n+1} < a_n$, and so the sequence is decreasing. Since all terms are positive, it is also bounded, and so it must converge. (Again it is possible to compute the limit; see Exercise 6.1.2.) ♣

Exercises for 6.1

Exercise 6.1.1 *Compute* $\lim_{x\to\infty} x^{1/x}$.

Exercise 6.1.2 *Use the squeeze theorem to show that* $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

Exercise 6.1.3 *Determine whether* $\{\sqrt{n+47} - \sqrt{n}\}_{n=0}^{\infty}$ *converges or diverges. If it converges, compute the limit.*

Exercise 6.1.4 *Determine whether* $\left\{\frac{n^2+1}{(n+1)^2}\right\}_{n=0}^{\infty}$ *converges or diverges. If it converges, compute the limit.*

Exercise 6.1.5 *Determine whether* $\left\{\frac{n+47}{\sqrt{n^2+3n}}\right\}_{n=1}^{\infty}$ *converges or diverges. If it converges, compute the limit.*

Exercise 6.1.6 *Determine whether* $\left\{\frac{2^n}{n!}\right\}_{n=0}^{\infty}$ *converges or diverges.*

6.2 Series

While much more can be said about sequences, we now turn to our principal interest, series. Recall that a series, roughly speaking, is the sum of a sequence: If $\{a_n\}_{n=0}^{\infty}$ is a sequence then the associated series is

$$\sum_{i=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots$$

Associated with a series is a second sequence, called the **s**equence of partial sums $\{s_n\}_{n=0}^{\infty}$:

$$s_n = \sum_{i=0}^{n} a_i.$$

So

$$s_0 = a_0, \quad s_1 = a_0 + a_1, \quad s_2 = a_0 + a_1 + a_2, \quad \ldots$$

A series converges if the sequence of partial sums converges, and otherwise the series diverges.

If $\{kx^n\}_{n=0}^{\infty}$ is a geometric sequence, then the associated series $\sum_{i=0}^{\infty} kx^i$ is called a geometric series.

Theorem 6.137: Geometric Series Convergence

If $|x| < 1$, *the geometric series* $\sum_i kx^i$ *converges to* $\frac{k}{1-x}$, *otherwise the series diverges (unless* $k = 0$*).*

Proof. If $a_n = kx^n$, $\sum_{n=0}^{\infty} a_n$ is called a **geometric series**. A typical partial sum is

$$s_n = k + kx + kx^2 + kx^3 + \cdots + kx^n = k(1 + x + x^2 + x^3 + \cdots + x^n).$$

We note that

$$\begin{aligned} s_n(1-x) &= k(1+x+x^2+x^3+\cdots+x^n)(1-x) \\ &= k(1+x+x^2+x^3+\cdots+x^n)1 - k(1+x+x^2+x^3+\cdots+x^{n-1}+x^n)x \\ &= k(1+x+x^2+x^3+\cdots+x^n-x-x^2-x^3-\cdots-x^n-x^{n+1}) \\ &= k(1-x^{n+1}) \end{aligned}$$

so

$$\begin{aligned} s_n(1-x) &= k(1-x^{n+1}) \\ s_n &= k\frac{1-x^{n+1}}{1-x}. \end{aligned}$$

If $|x| < 1$, $\lim_{n\to\infty} x^n = 0$ so

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} k\frac{1-x^{n+1}}{1-x} = k\frac{1}{1-x}.$$

Thus, when $|x| < 1$ the geometric series converges to $k/(1-x)$. ♣

When, for example, $k = 1$ and $x = 1/2$:

$$s_n = \frac{1-(1/2)^{n+1}}{1-1/2} = \frac{2^{n+1}-1}{2^n} = 2 - \frac{1}{2^n} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-1/2} = 2.$$

We began the chapter with the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n},$$

namely, the geometric series without the first term 1. Each partial sum of this series is 1 less than the corresponding partial sum for the geometric series, so of course the limit is also one less than the value of the geometric series, that is,

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

It is not hard to see that the following theorem follows from Theorem 6.123.

Theorem 6.138: Series are Linear

Suppose that $\sum a_n$ and $\sum b_n$ are convergent series, and c is a constant. Then

1. *$\sum c a_n$ is convergent and $\sum c a_n = c \sum a_n$*
2. *$\sum (a_n + b_n)$ is convergent and $\sum (a_n + b_n) = \sum a_n + \sum b_n$.*

Note that when c is non-zero, the converse of the first part of this theorem is also true. That is, if $\sum c a_n$ is convergent, then $\sum a_n$ is also convergent; if $\sum c a_n$ converges then $\frac{1}{c} \sum c a_n$ must converge.

On the other hand, the converse of the second part of the theorem is not true. For example, if $a_n = 1$ and $b_n = -1$, then $\sum a_n + \sum b_n = \sum 0 = 0$ converges, but each of $\sum a_n$ and $\sum b_n$ diverges.

In general, the sequence of partial sums s_n is harder to understand and analyze than the sequence of terms a_n, and it is difficult to determine whether series converge and if so to what. The following result will let us deal with some simple cases easily.

Theorem 6.139: Divergence Test

If $\sum a_n$ *converges then* $\lim_{n\to\infty} a_n = 0$.

Proof. Since $\sum a_n$ converges, $\lim_{n\to\infty} s_n = L$ and $\lim_{n\to\infty} s_{n-1} = L$, because this really says the same thing but "renumbers" the terms. By Theorem 6.123,

$$\lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = L - L = 0.$$

But

$$s_n - s_{n-1} = (a_0 + a_1 + a_2 + \cdots + a_n) - (a_0 + a_1 + a_2 + \cdots + a_{n-1}) = a_n,$$

so as desired $\lim_{n\to\infty} a_n = 0$. ♣

This theorem presents an easy divergence test: Given a series $\sum a_n$, if the limit $\lim_{n\to\infty} a_n$ does not exist or has a value other than zero, the series diverges. Note well that the converse is *not* true: If $\lim_{n\to\infty} a_n = 0$ then the series does not necessarily converge.

Theorem 6.140: The n-th Term Test

If $\lim_{n\to\infty} a_n \neq 0$ *or if the limit does not exist, then* $\sum a_n$ *diverges.*

Proof. Consider the statement of the theorem in contrapositive form:

$$\textbf{If } \sum_{n=1}^{\infty} a_n \text{ converges, then } \lim_{n\to\infty} a_n = 0.$$

If s_n are the partial sums of the series, then the assumption that the series converges gives us

$$\lim_{n\to\infty} s_n = s$$

for some number s. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0.$$

♣

Example 6.141:

Show that $\sum_{n=1}^{\infty} \frac{n}{n+1}$ *diverges.*

Solution. We compute the limit:

$$\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0.$$

Looking at the first few terms perhaps makes it clear that the series has no chance of converging:

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots$$

will just get larger and larger; indeed, after a bit longer the series starts to look very much like $\cdots + 1 + 1 + 1 + 1 + \cdots$, and of course if we add up enough 1's we can make the sum as large as we desire. ♣

Example 6.142: Harmonic Series

Show that $\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges.*

Solution. Here the theorem does not apply: $\lim_{n\to\infty} 1/n = 0$, so it looks like perhaps the series converges. Indeed, if you have the fortitude (or the software) to add up the first 1000 terms you will find that

$$\sum_{n=1}^{1000} \frac{1}{n} \approx 7.49,$$

so it might be reasonable to speculate that the series converges to something in the neighborhood of 10. But in fact the partial sums do go to infinity; they just get big very, very slowly. Consider the following:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{1}{2} + \frac{1}{2}$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$$

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{16} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{16} = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$$

and so on. By swallowing up more and more terms we can always manage to add at least another $1/2$ to the sum, and by adding enough of these we can make the partial sums as big as we like. In fact, it's not hard to see from this pattern that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n} > 1 + \frac{n}{2},$$

so to make sure the sum is over 100, for example, we'd add up terms until we get to around $1/2^{198}$, that is, about $4 \cdot 10^{59}$ terms. This series, $\sum(1/n)$, is called the **harmonic series**. ♣

We will often make use of the fact that the first few (e.g. any finite number of) terms in a series are irrelevant when determining whether it will converge. In other words, $\sum_{n=0}^{\infty} a_n$ converges if and only if $\sum_{n=N}^{\infty} a_n$ converges for some $N \geq 1$.

Exercises for 6.2

Exercise 6.2.1 *Explain why* $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2n^2+1}$ *diverges.*

Exercise 6.2.2 *Explain why* $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2^{1/n}+14}$ *diverges.*

Exercise 6.2.3 *Explain why* $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n}$ *diverges.*

Exercise 6.2.4 *Compute* $\displaystyle\sum_{n=0}^{\infty} \frac{4}{(-3)^n} - \frac{3}{3^n}$.

Exercise 6.2.5 *Compute* $\displaystyle\sum_{n=0}^{\infty} \frac{3}{2^n} + \frac{4}{5^n}$.

Exercise 6.2.6 *Compute* $\displaystyle\sum_{n=0}^{\infty} \frac{4^{n+1}}{5^n}$.

Exercise 6.2.7 *Compute* $\displaystyle\sum_{n=0}^{\infty} \frac{3^{n+1}}{7^{n+1}}$.

Exercise 6.2.8 *Compute* $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n$.

Exercise 6.2.9 *Compute* $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{5^{n+1}}$.

6.3 The Integral Test

It is generally quite difficult, often impossible, to determine the value of a series exactly. In many cases it is possible at least to determine whether or not the series converges, and so we will spend most of our time on this problem.

If all of the terms a_n in a series are non-negative, then clearly the sequence of partial sums s_n is non-decreasing. This means that if we can show that the sequence of partial sums is bounded, the series must converge. Many useful and interesting series have this property, and they are among the easiest to understand. Let's look at an example.

Example 6.143:

Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converges.*

Solution. The terms $1/n^2$ are positive and decreasing, and since $\lim_{x\to\infty} 1/x^2 = 0$, the terms $1/n^2$ approach zero. We seek an upper bound for all the partial sums, that is, we want to find a number N so that $s_n \le N$ for every n. The upper bound is provided courtesy of integration, and is illustrated in figure 6.40.

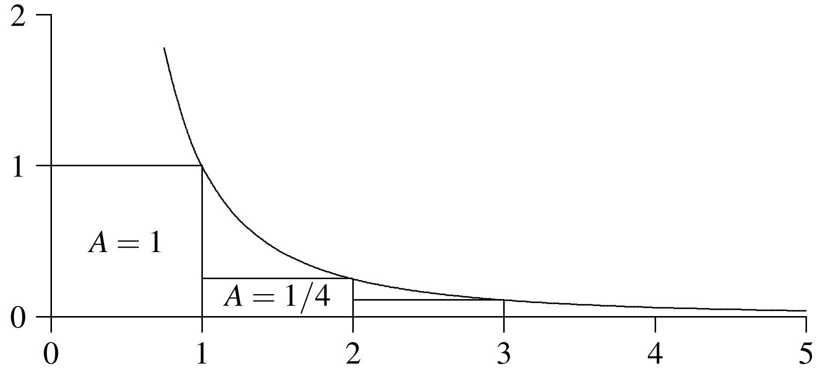


Figure 6.40: Graph of $y = 1/x^2$ with rectangles.

The figure shows the graph of $y = 1/x^2$ together with some rectangles that lie completely below the curve and that all have base length one. Because the heights of the rectangles are determined by the height of the curve, the areas of the rectangles are $1/1^2$, $1/2^2$, $1/3^2$, and so on—in other words, exactly the terms of the series. The partial sum s_n is simply the sum of the areas of the first n rectangles. Because the rectangles all lie between the curve and the x-axis, any sum of rectangle areas is less than the corresponding area under the curve, and so of course any sum of rectangle areas is less than the area under the entire curve. Unfortunately, because of the asymptote at $x = 0$, the integral $\int_0^\infty \frac{1}{x^2}$ is infinite, but we can deal with this by separating the first term from the series and integrating from 1:

$$s_n = \sum_{i=1}^{n} \frac{1}{i^2} = 1 + \sum_{i=2}^{n} \frac{1}{i^2} < 1 + \int_1^n \frac{1}{x^2}\,dx < 1 + \int_1^\infty \frac{1}{x^2}\,dx = 1 + 1 = 2$$

(Recalling that we computed this improper integral in section 3.7). Since the sequence of partial sums s_n is increasing and bounded above by 2, we know that $\lim_{n\to\infty} s_n = L < 2$, and so the series converges to some number less than 2. In fact, it is possible, though difficult, to show that $L = \pi^2/6 \approx 1.6$. ♣

We already know that $\sum 1/n$ diverges. What goes wrong if we try to apply this technique to it? Here's the calculation:

$$s_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < 1 + \int_1^n \frac{1}{x}\,dx < 1 + \int_1^\infty \frac{1}{x}\,dx = 1 + \infty.$$

The problem is that the improper integral doesn't converge. Note that this does *not* prove that $\sum 1/n$ diverges, just that this particular calculation fails to prove that it converges. A slight modification, however, allows us to prove in a second way that $\sum 1/n$ diverges.

Consider a slightly altered version of Figure 6.40, shown in Figure 6.41.

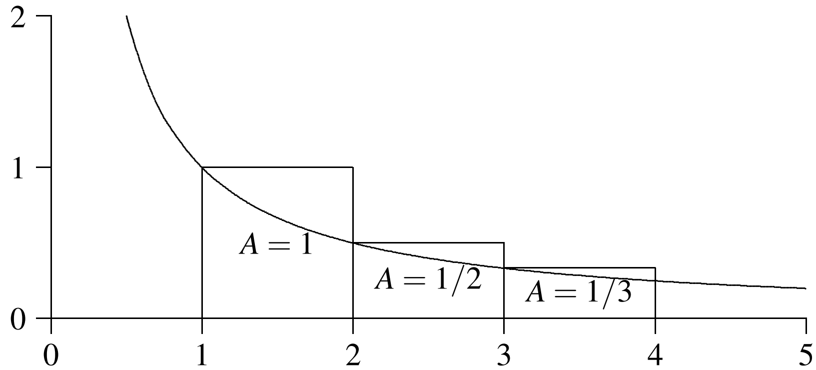


Figure 6.41: Graph of $y = 1/x$ with rectangles.

This time the rectangles are above the curve, that is, each rectangle completely contains the corresponding area under the curve. This means that

$$s_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > \int_1^{n+1} \frac{1}{x}\, dx = \ln x \Big|_1^{n+1} = \ln(n+1).$$

As n gets bigger, $\ln(n+1)$ goes to infinity, so the sequence of partial sums s_n must also go to infinity, so the harmonic series diverges.

The key fact in this example is that

$$\lim_{n\to\infty} \int_1^{n+1} \frac{1}{x}\, dx = \int_1^{\infty} \frac{1}{x}\, dx = \infty$$

So these two examples taken together indicate that we can prove that a series converges or prove that it diverges with a single calculation of an improper integral. This is known as the **integral test**, which we state as a theorem.

Theorem 6.144: Integral Test

Suppose that $f(x) > 0$ and is decreasing on the infinite interval $[1,\infty)$ and that $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the improper integral $\int_1^{\infty} f(x)\, dx$ converges.

The two examples we have seen are called ***p*-series**; a p-series is any series of the form $\sum 1/n^p$. If $p \leq 0$, $\lim_{n\to\infty} 1/n^p \neq 0$, so the series diverges. For positive values of p we can determine precisely which series converge.

Theorem 6.145: p-Series Convergence

A p-series with $p > 0$ converges if and only if $p > 1$.

Proof. We use the integral test; we have already done $p = 1$, so assume that $p \neq 1$.

$$\int_1^{\infty} \frac{1}{x^p}\, dx = \lim_{D\to\infty} \frac{x^{1-p}}{1-p}\Big|_1^D = \lim_{D\to\infty} \frac{D^{1-p}}{1-p} - \frac{1}{1-p}.$$

If $p > 1$ then $1-p < 0$ and $\lim_{D\to\infty} D^{1-p} = 0$, so the integral converges. If $0 < p < 1$ then $1-p > 0$ and $\lim_{D\to\infty} D^{1-p} = \infty$, so the integral diverges. ♣

Example 6.146: *p*-Series

Show that $\sum_{n=1}^{\infty} \frac{1}{n^3}$ *converges.*

Solution. We could of course use the integral test, but now that we have the theorem we may simply note that this is a p-series with $p > 1$. ♣

Example 6.147: *p*-Series

Show that $\sum_{n=1}^{\infty} \frac{5}{n^4}$ *converges.*

Solution. We know that if $\sum_{n=1}^{\infty} 1/n^4$ converges then $\sum_{n=1}^{\infty} 5/n^4$ also converges, by Theorem 6.138. Since $\sum_{n=1}^{\infty} 1/n^4$ is a convergent p-series, $\sum_{n=1}^{\infty} 5/n^4$ converges also. ♣

Example 6.148: *p*-Series

Show that $\sum_{n=1}^{\infty} \frac{5}{\sqrt{n}}$ *diverges.*

Solution. This also follows from Theorem 6.138: Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a p-series with $p = 1/2 < 1$, it diverges, and so does $\sum_{n=1}^{\infty} \frac{5}{\sqrt{n}}$. ♣

Since it is typically difficult to compute the value of a series exactly, a good approximation is frequently required. In a real sense, a good approximation is only as good as we know it is, that is, while an approximation may in fact be good, it is only valuable in practice if we can guarantee its accuracy to some degree. This guarantee is usually easy to come by for series with decreasing positive terms.

Example 6.149:

Approximate $\sum 1/n^2$ *to within 0.01.*

Solution. Referring to Figure 6.40, if we approximate the sum by $\sum_{n=1}^{N} 1/n^2$, the size of the error we make is the total area of the remaining rectangles, all of which lie under the curve $1/x^2$ from $x = N$ to infinity. So we know the true value of the series is larger than the approximation, and no bigger than the approximation plus the area under the curve from N to infinity. Roughly, then, we need to find N so that

$$\int_N^{\infty} \frac{1}{x^2}\,dx < 1/100.$$

We can compute the integral:

$$\int_N^\infty \frac{1}{x^2}\,dx = \frac{1}{N},$$

so if we choose $N = 100$ the error will be less than 0.01. Adding up the first 100 terms gives approximately 1.634983900. In fact, we can do a bit better. Since we know that the correct value is between our approximation and our approximation plus the error (not minus), we can cut our error bound in half by taking the value midway between these two values. If we take $N = 50$, we get a sum of 1.6251327 with an error of at most 0.02, so the correct value is between 1.6251327 and 1.6451327, and therefore the value halfway between these, 1.6351327, is within 0.01 of the correct value. We have mentioned that the true value of this series can be shown to be $\pi^2/6 \approx 1.644934068$ which is 0.0098 more than our approximation, and so (just barely) within the required error. Frequently approximations will be even better than the "guaranteed" accuracy, but not always, as this example demonstrates. ♣

Exercises for 6.3

Determine whether each series converges or diverges.

Exercise 6.3.1 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\pi/4}}$

Exercise 6.3.2 $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

Exercise 6.3.3 $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

Exercise 6.3.4 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Exercise 6.3.5 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n}$

Exercise 6.3.6 $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^n}$

Exercise 6.3.7 $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln n}$

Exercise 6.3.8 $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

Exercise 6.3.9 *Find an N so that* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ *is between* $\displaystyle\sum_{n=1}^{N} \frac{1}{n^4}$ *and* $\displaystyle\sum_{n=1}^{N} \frac{1}{n^4} + 0.005$.

Exercise 6.3.10 *Find an N so that* $\displaystyle\sum_{n=0}^{\infty} \frac{1}{e^n}$ *is between* $\displaystyle\sum_{n=0}^{N} \frac{1}{e^n}$ *and* $\displaystyle\sum_{n=0}^{N} \frac{1}{e^n} + 10^{-4}$.

Exercise 6.3.11 *Find an N so that* $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ *is between* $\displaystyle\sum_{n=1}^{N} \frac{\ln n}{n^2}$ *and* $\displaystyle\sum_{n=1}^{N} \frac{\ln n}{n^2} + 0.005$.

Exercise 6.3.12 *Find an N so that* $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ *is between* $\displaystyle\sum_{n=2}^{N} \frac{1}{n(\ln n)^2}$ *and* $\displaystyle\sum_{n=2}^{N} \frac{1}{n(\ln n)^2} + 0.005$.

6.4 Alternating Series

Next we consider series with both positive and negative terms, but in a regular pattern: they alternate, as in the **alternating harmonic series**:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \frac{1}{1} + \frac{-1}{2} + \frac{1}{3} + \frac{-1}{4} + \cdots = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.$$

In this example the magnitude of the terms decrease, that is, $|a_n|$ forms a decreasing sequence, although this is not required in an alternating series. Recall that for a series with positive terms, if the limit of the terms is not zero, the series cannot converge; but even if the limit of the terms is zero, the series still may not converge. It turns out that for alternating series, the series converges exactly when the limit of the terms is zero. In Figure 6.42, we illustrate what happens to the partial sums of the alternating harmonic series. Because the sizes of the terms a_n are decreasing, the odd partial sums s_1, s_3, s_5, and so on, form a decreasing sequence that is bounded below by s_2, so this sequence must converge. Likewise, the even partial sums s_2, s_4, s_6, and so on, form an increasing sequence that is bounded above by s_1, so this sequence also converges. Since all the even numbered partial sums are less than all the odd numbered ones, and since the "jumps" (that is, the a_i terms) are getting smaller and smaller, the two sequences must converge to the same value, meaning the entire sequence of partial sums $s_1, s_2, s_3, \ldots$ converges as well.

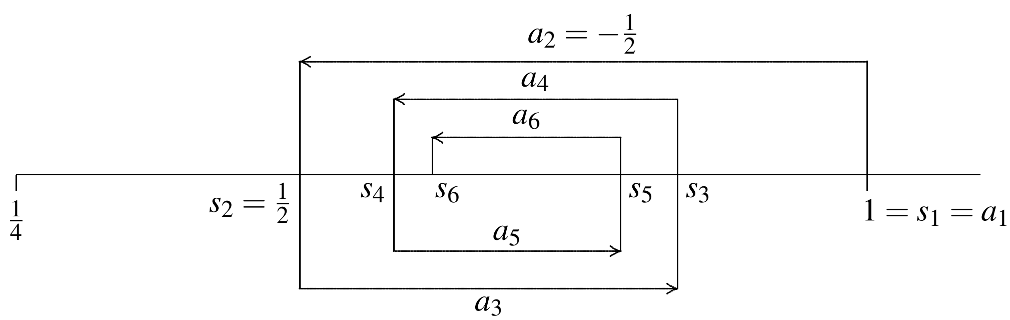


Figure 6.42: The alternating harmonic series.

The same argument works for any alternating sequence with terms that decrease in absolute value. The alternating series test is worth calling a theorem.

Theorem 6.150: Alternating Series Test

Suppose that $\{a_n\}_{n=1}^{\infty}$ *is a non-increasing sequence of positive numbers and* $\lim_{n\to\infty} a_n = 0$*. Then the alternating series* $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ *converges.*

Proof. The odd-numbered partial sums, $s_1, s_3, s_5, \ldots, s_{2k+1}, \ldots$, form a non-increasing sequence, because $s_{2k+3} = s_{2k+1} - a_{2k+2} + a_{2k+3} \leq s_{2k+1}$, since $a_{2k+2} \geq a_{2k+3}$. This sequence is bounded below by s_2, so it must converge, to some value L. Likewise, the partial sums $s_2, s_4, s_6, \ldots, s_{2k}, \ldots$, form a non-decreasing sequence that is bounded above by s_1, so this sequence also converges, to some value M. Since $\lim_{n\to\infty} a_n = 0$ and $s_{2k+1} = s_{2k} + a_{2k+1}$,

$$L = \lim_{k\to\infty} s_{2k+1} = \lim_{k\to\infty}(s_{2k} + a_{2k+1}) = \lim_{k\to\infty} s_{2k} + \lim_{k\to\infty} a_{2k+1} = M + 0 = M,$$

so $L = M$; the two sequences of partial sums converge to the same limit, and this means the entire sequence of partial sums also converges to L. ♣

Another useful fact is implicit in this discussion. Suppose that

$$L = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

and that we approximate L by a finite part of this sum, say

$$L \approx \sum_{n=1}^{N} (-1)^{n-1} a_n.$$

Because the terms are decreasing in size, we know that the true value of L must be between this approximation and the next one, that is, between

$$\sum_{n=1}^{N} (-1)^{n-1} a_n \quad \text{and} \quad \sum_{n=1}^{N+1} (-1)^{n-1} a_n.$$

Depending on whether N is odd or even, the second will be larger or smaller than the first.

Example 6.151:

Approximate the sum of the alternating harmonic series to within 0.05.

Solution. We need to go to the point at which the next term to be added or subtracted is $1/10$. Adding up the first nine and the first ten terms we get approximately 0.746 and 0.646. These are $1/10$ apart, so the value halfway between them, 0.696, is within 0.05 of the correct value. ♣

We have considered alternating series with first index 1, and in which the first term is positive, but a little thought shows this is not crucial. The same test applies to any similar series, such as $\sum_{n=0}^{\infty} (-1)^n a_n$, $\sum_{n=1}^{\infty} (-1)^n a_n$, $\sum_{n=17}^{\infty} (-1)^n a_n$, etc.

Exercises for 6.4

Determine whether the following series converge or diverge.

Exercise 6.4.1 $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+5}$

Exercise 6.4.2 $\displaystyle\sum_{n=4}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n-3}}$

Exercise 6.4.3 $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{3n-2}$

Exercise 6.4.4 $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\ln n}{n}$

Exercise 6.4.5 *Approximate* $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^3}$ *to within 0.005.*

Exercise 6.4.6 *Approximate* $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^4}$ *to within 0.005.*

6.5 Comparison Tests

As we begin to compile a list of convergent and divergent series, new ones can sometimes be analyzed by comparing them to ones that we already understand.

Example 6.152:

Does $\sum_{n=2}^{\infty} \frac{1}{n^2 \ln n}$ *converge?*

Solution. The obvious first approach, based on what we know, is the integral test. Unfortunately, we can't compute the required antiderivative. But looking at the series, it would appear that it must converge, because the terms we are adding are smaller than the terms of a p-series, that is,

$$\frac{1}{n^2 \ln n} < \frac{1}{n^2},$$

when $n \geq 3$. Since adding up the terms $1/n^2$ doesn't get "too big", the new series "should" also converge. Let's make this more precise.

The series $\sum_{n=2}^{\infty} \frac{1}{n^2 \ln n}$ converges if and only if $\sum_{n=3}^{\infty} \frac{1}{n^2 \ln n}$ converges—all we've done is dropped the initial term. We know that $\sum_{n=3}^{\infty} \frac{1}{n^2}$ converges. Looking at two typical partial sums:

$$s_n = \frac{1}{3^2 \ln 3} + \frac{1}{4^2 \ln 4} + \frac{1}{5^2 \ln 5} + \cdots + \frac{1}{n^2 \ln n} < \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots + \frac{1}{n^2} = t_n.$$

Since the p-series converges, say to L, and since the terms are positive, $t_n < L$. Since the terms of the new series are positive, the s_n form an increasing sequence and $s_n < t_n < L$ for all n. Hence the sequence $\{s_n\}$ is bounded and so converges. ♣

Sometimes, even when the integral test applies, comparison to a known series is easier, so it's generally a good idea to think about doing a comparison before doing the integral test.

Example 6.153:

Does $\sum_{n=2}^{\infty} \frac{|\sin n|}{n^2}$ *converge?*

Solution. We can't apply the integral test here, because the terms of this series are not decreasing. Just as in the previous example, however,

$$\frac{|\sin n|}{n^2} \leq \frac{1}{n^2},$$

because $|\sin n| \leq 1$. Once again the partial sums are non-decreasing and bounded above by $\sum 1/n^2 = L$, so the new series converges. ♣

Like the integral test, the comparison test can be used to show both convergence and divergence. In the case of the integral test, a single calculation will confirm whichever is the case. To use the comparison test we must first have a good idea as to convergence or divergence and pick the sequence for comparison accordingly.

Example 6.154:

Does $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-3}}$ *converge?*

Solution. We observe that the -3 should have little effect compared to the n^2 inside the square root, and therefore guess that the terms are enough like $1/\sqrt{n^2} = 1/n$ that the series should diverge. We attempt to show this by comparison to the harmonic series. We note that

$$\frac{1}{\sqrt{n^2-3}} > \frac{1}{\sqrt{n^2}} = \frac{1}{n},$$

so that

$$s_n = \frac{1}{\sqrt{2^2-3}} + \frac{1}{\sqrt{3^2-3}} + \cdots + \frac{1}{\sqrt{n^2-3}} > \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = t_n,$$

where t_n is 1 less than the corresponding partial sum of the harmonic series (because we start at $n = 2$ instead of $n = 1$). Since $\lim_{n\to\infty} t_n = \infty$, $\lim_{n\to\infty} s_n = \infty$ as well. ♣

So the general approach is this: If you believe that a new series is convergent, attempt to find a convergent series whose terms are larger than the terms of the new series; if you believe that a new series is divergent, attempt to find a divergent series whose terms are smaller than the terms of the new series.

Example 6.155:

Does $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+3}}$ *converge?*

Solution. Just as in the last example, we guess that this is very much like the harmonic series and so diverges. Unfortunately,

$$\frac{1}{\sqrt{n^2+3}} < \frac{1}{n},$$

so we can't compare the series directly to the harmonic series. A little thought leads us to

$$\frac{1}{\sqrt{n^2+3}} > \frac{1}{\sqrt{n^2+3n^2}} = \frac{1}{2n},$$

so if $\sum 1/(2n)$ diverges then the given series diverges. But since $\sum 1/(2n) = (1/2)\sum 1/n$, Theorem 6.138 implies that it does indeed diverge. ♣

For reference we summarize the comparison test in a theorem.

Theorem 6.156: Comparison Theorem

Suppose that a_n and b_n are non-negative for all n and that $a_n \le b_n$ when $n \ge N$, for some N.

- *If $\sum_{n=0}^{\infty} b_n$ converges, so does $\sum_{n=0}^{\infty} a_n$.*
- *If $\sum_{n=0}^{\infty} a_n$ diverges, so does $\sum_{n=0}^{\infty} b_n$.*

Exercises for 6.5

Determine whether the series converge or diverge.

Exercise 6.5.1 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^2+3n+5}$

Exercise 6.5.2 $\displaystyle\sum_{n=2}^{\infty} \frac{1}{2n^2+3n-5}$

Exercise 6.5.3 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^2-3n-5}$

Exercise 6.5.4 $\displaystyle\sum_{n=1}^{\infty} \frac{3n+4}{2n^2+3n+5}$

Exercise 6.5.5 $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+4}{2n^2+3n+5}$

Exercise 6.5.6 $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$

Exercise 6.5.7 $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

Exercise 6.5.8 $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$

Exercise 6.5.9 $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{2^n+5^n}$

Exercise 6.5.10 $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{2^n+3^n}$

6.6 Absolute Convergence

Roughly speaking there are two ways for a series to converge: As in the case of $\sum 1/n^2$, the individual terms get small very quickly, so that the sum of all of them stays finite, or, as in the case of $\sum(-1)^{n-1}/n$, the terms don't get small fast enough ($\sum 1/n$ diverges), but a mixture of positive and negative terms provides enough cancellation to keep the sum finite. You might guess from what we've seen that if the terms get small fast enough that the sum of their absolute values converges, then the series will still converge regardless of which terms are actually positive or negative.

Theorem 6.157: Absolute Convergence

If $\displaystyle\sum_{n=0}^{\infty} |a_n|$ *converges, then* $\displaystyle\sum_{n=0}^{\infty} a_n$ *converges.*

Proof. Note that $0 \leq a_n + |a_n| \leq 2|a_n|$ so by the comparison test $\displaystyle\sum_{n=0}^{\infty} (a_n + |a_n|)$ converges. Now

$$\sum_{n=0}^{\infty} (a_n + |a_n|) - \sum_{n=0}^{\infty} |a_n| = \sum_{n=0}^{\infty} a_n + |a_n| - |a_n| = \sum_{n=0}^{\infty} a_n$$

converges by Theorem 6.138. ♣

So given a series $\sum a_n$ with both positive and negative terms, you should first ask whether $\sum |a_n|$ converges. This may be an easier question to answer, because we have tests that apply specifically to terms with non-negative terms.

If $\sum |a_n|$ converges then you know that $\sum a_n$ converges as well. If $\sum |a_n|$ diverges then it still may be true that $\sum a_n$ converges, but you will need to use other techniques to decide. Intuitively this results says that it is (potentially) easier for $\sum a_n$ to converge than for $\sum |a_n|$ to converge, because terms may partially cancel in the first series.

If $\sum |a_n|$ converges we say that $\sum a_n$ converges **absolutely**; to say that $\sum a_n$ converges absolutely is to say that the terms of the series get small (in absolute value) quickly enough to guarantee that the series converges, regardless of whether any of the terms cancel each other. If $\sum a_n$ converges but $\sum |a_n|$ does not, we say that $\sum a_n$ converges **conditionally**. For example $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2}$ converges absolutely, while $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ converges conditionally.

Example 6.158:

Does $\sum_{n=2}^{\infty}\frac{\sin n}{n^2}$ *converge?*

Solution. In Example 6.153 we saw that $\sum_{n=2}^{\infty}\frac{|\sin n|}{n^2}$ converges, so the given series converges absolutely. ♣

Example 6.159:

Does $\sum_{n=0}^{\infty}(-1)^n\frac{3n+4}{2n^2+3n+5}$ *converge?*

Solution. Taking the absolute value, $\sum_{n=0}^{\infty}\frac{3n+4}{2n^2+3n+5}$ diverges by comparison to $\sum_{n=1}^{\infty}\frac{3}{10n}$, so if the series converges it does so conditionally. It is true that $\lim_{n\to\infty}(3n+4)/(2n^2+3n+5)=0$, so to apply the alternating series test we need to know whether the terms are decreasing. If we let $f(x)=(3x+4)/(2x^2+3x+5)$ then $f'(x)=-(6x^2+16x-3)/(2x^2+3x+5)^2$, and it is not hard to see that this is negative for $x\geq 1$, so the series is decreasing and by the alternating series test it converges. ♣

Exercises for 6.6

Determine whether each series converges absolutely, converges conditionally, or diverges.

Exercise 6.6.1 $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{2n^2+3n+5}$

Exercise 6.6.2 $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3n^2+4}{2n^2+3n+5}$

Exercise 6.6.3 $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\ln n}{n}$

Exercise 6.6.4 $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\ln n}{n^3}$

Exercise 6.6.5 $\sum_{n=2}^{\infty}(-1)^{n}\frac{1}{\ln n}$

Exercise 6.6.6 $\sum_{n=0}^{\infty}(-1)^{n}\frac{3^n}{2^n+5^n}$

Exercise 6.6.7 $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{3^n}{2^n+3^n}$

Exercise 6.6.8 $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\arctan n}{n}$

6.7 The Ratio and Root Tests

Does the series $\displaystyle\sum_{n=0}^{\infty}\frac{n^5}{5^n}$ converge? It is possible, but a bit unpleasant, to approach this with the integral test or the comparison test, but there is an easier way. Consider what happens as we move from one term to the next in this series:

$$\cdots+\frac{n^5}{5^n}+\frac{(n+1)^5}{5^{n+1}}+\cdots$$

The denominator goes up by a factor of 5, $5^{n+1}=5\cdot 5^n$, but the numerator goes up by much less: $(n+1)^5=n^5+5n^4+10n^3+10n^2+5n+1$, which is much less than $5n^5$ when n is large, because $5n^4$ is much less than n^5. So we might guess that in the long run it begins to look as if each term is $1/5$ of the previous term. We have seen series that behave like this: The geometric series.

$$\sum_{n=0}^{\infty}\frac{1}{5^n}=\frac{5}{4},$$

So we might try comparing the given series to some variation of this geometric series. This is possible, but a bit messy. We can in effect do the same thing, but bypass most of the unpleasant work.

The key is to notice that

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)^5}{5^{n+1}}\frac{5^n}{n^5}=\lim_{n\to\infty}\frac{(n+1)^5}{n^5}\frac{1}{5}=1\cdot\frac{1}{5}=\frac{1}{5}.$$

This is really just what we noticed above, done a bit more formally: in the long run, each term is one fifth of the previous term. Now pick some number between $1/5$ and 1, say $1/2$. Because

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\frac{1}{5},$$

then when n is big enough, say $n\geq N$ for some N,

$$\frac{a_{n+1}}{a_n}<\frac{1}{2}\quad\text{so}\quad a_{n+1}<\frac{a_n}{2}.$$

So $a_{N+1}<a_N/2$, $a_{N+2}<a_{N+1}/2<a_N/4$, $a_{N+3}<a_{N+2}/2<a_N/8$, and so on. The general form is $a_{N+k}<a_N/2^k$. So if we look at the series

$$\sum_{k=0}^{\infty}a_{N+k}=a_N+a_{N+1}+a_{N+2}+a_{N+3}+\cdots+a_{N+k}+\cdots,$$

its terms are less than or equal to the terms of the sequence

$$a_N+\frac{a_N}{2}+\frac{a_N}{4}+\frac{a_N}{8}+\cdots+\frac{a_N}{2^k}+\cdots=\sum_{k=0}^{\infty}\frac{a_N}{2^k}=2a_N.$$

So by the comparison test, $\sum_{k=0}^{\infty} a_{N+k}$ converges, and this means that $\sum_{n=0}^{\infty} a_n$ converges, since we've just added the fixed number $a_0 + a_1 + \cdots + a_{N-1}$.

Under what circumstances could we do this? What was crucial was that the limit of a_{n+1}/a_n, say L, was less than 1 so that we could pick a value r so that $L < r < 1$. The fact that $L < r$ ($1/5 < 1/2$ in our example) means that we can compare the series $\sum a_n$ to $\sum r^n$, and the fact that $r < 1$ guarantees that $\sum r^n$ converges. That's really all that is required to make the argument work. We also made use of the fact that the terms of the series were positive; in general we simply consider the absolute values of the terms and we end up testing for absolute convergence.

Theorem 6.160: The Ratio Test

Suppose that $\lim_{n\to\infty} |a_{n+1}/a_n| = L$. *If* $L < 1$ *the series* $\sum a_n$ *converges absolutely, if* $L > 1$ *the series diverges, and if* $L = 1$ *this test gives no information.*

Proof. The example above essentially proves the first part of this, if we simply replace $1/5$ by L and $1/2$ by r. Suppose that $L > 1$, and pick r so that $1 < r < L$. Then for $n \geq N$, for some N,

$$\frac{|a_{n+1}|}{|a_n|} > r \quad \text{and} \quad |a_{n+1}| > r|a_n|.$$

This implies that $|a_{N+k}| > r^k |a_N|$, but since $r > 1$ this means that $\lim_{k\to\infty} |a_{N+k}| \neq 0$, which means also that $\lim_{n\to\infty} a_n \neq 0$. By the divergence test, the series diverges.

To see that we get no information when $L = 1$, we need to exhibit two series with $L = 1$, one that converges and one that diverges. The series $\sum 1/n^2$ and $\sum 1/n$ provide a simple example. ♣

The ratio test is particularly useful for series involving the factorial function.

Example 6.161:

Analyze $\sum_{n=0}^{\infty} \frac{5^n}{n!}$.

Solution.

$$\lim_{n\to\infty} \frac{5^{n+1}}{(n+1)!} \frac{n!}{5^n} = \lim_{n\to\infty} \frac{5^{n+1}}{5^n} \frac{n!}{(n+1)!} = \lim_{n\to\infty} 5 \frac{1}{(n+1)} = 0.$$

Since $0 < 1$, the series converges. ♣

A similar argument justifies a similar test that is occasionally easier to apply.

Theorem 6.162: The Root Test

Suppose that $\lim_{n\to\infty} |a_n|^{1/n} = L$. *If* $L < 1$ *the series* $\sum a_n$ *converges absolutely, if* $L > 1$ *the series diverges, and if* $L = 1$ *this test gives no information.*

The proof of the root test is actually easier than that of the ratio test, and is left as an exercise.

Example 6.163:

Analyze $\sum_{n=0}^{\infty} \frac{5^n}{n^n}$.

Solution. The ratio test turns out to be a bit difficult on this series (try it). Using the root test:

$$\lim_{n\to\infty}\left(\frac{5^n}{n^n}\right)^{1/n} = \lim_{n\to\infty}\frac{(5^n)^{1/n}}{(n^n)^{1/n}} = \lim_{n\to\infty}\frac{5}{n} = 0.$$

Since $0 < 1$, the series converges. ♣

The root test is frequently useful when n appears as an exponent in the general term of the series.

Exercises for 6.7

Exercise 6.7.1 *Compute* $\lim_{n\to\infty} |a_{n+1}/a_n|$ *for the series* $\sum 1/n^2$.

Exercise 6.7.2 *Compute* $\lim_{n\to\infty} |a_{n+1}/a_n|$ *for the series* $\sum 1/n$.

Exercise 6.7.3 *Compute* $\lim_{n\to\infty} |a_n|^{1/n}$ *for the series* $\sum 1/n^2$.

Exercise 6.7.4 *Compute* $\lim_{n\to\infty} |a_n|^{1/n}$ *for the series* $\sum 1/n$.

Exercise 6.7.5 *Determine whether the series converge.*

(a) $\sum_{n=0}^{\infty} (-1)^n \frac{3^n}{5^n}$

(b) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

(c) $\sum_{n=1}^{\infty} \frac{n^5}{n^n}$

(d) $\sum_{n=1}^{\infty} \frac{(n!)^2}{n^n}$

Exercise 6.7.6 *Prove Theorem 6.162, the root test.*

6.8 Power Series

Recall that the sum of a geometric series can be expressed using the simple formula:

$$\sum_{n=0}^{\infty} kx^n = \frac{k}{1-x},$$

if $|x| < 1$, and that the series diverges when $|x| \geq 1$. At the time, we thought of x as an unspecified constant, but we could just as well think of it as a variable, in which case the series

$$\sum_{n=0}^{\infty} kx^n$$

is a function, namely, the function $k/(1-x)$, as long as $|x| < 1$: Looking at this from the opposite perspective, this means that the function $k/(1-x)$ can be represented as the sum of an infinite series. Why would this be useful? While $k/(1-x)$ is a reasonably easy function to deal with, the more complicated representation $\sum kx^n$ does have some advantages: it appears to be an infinite version of one of the simplest function types—a polynomial. Later on we will investigate some of the ways we can take advantage of this 'infinite polynomial' representation, but first we should ask if other functions can even be represented this way.

The geometric series has a special feature that makes it unlike a typical polynomial—the coefficients of the powers of x are all the same, namely k. We will need to allow more general coefficients if we are to get anything other than the geometric series.

Definition 6.164: Power Series

A power series is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n,$$

where each a_n is a real number.

As we did in the section on sequences, we can think of the a_n as being a function $a(n)$ defined on the non-negative integers. Note, however, that the a_n do not depend on x.

Example 6.165:

Determine whether the power series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$ *converges.*

Solution. We can investigate convergence using the ratio test:

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{n+1}\frac{n}{|x|^n} = \lim_{n\to\infty} |x|\frac{n}{n+1} = |x|.$$

Thus when $|x| < 1$ the series converges and when $|x| > 1$ it diverges, leaving only two values in doubt. When $x = 1$ the series is the harmonic series and diverges; when $x = -1$ it is the alternating harmonic series (actually the negative of the usual alternating harmonic series) and converges. Thus, we may think of $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$ as a function from the interval $[-1,1)$ to the real numbers. ♣

A bit of thought reveals that the ratio test applied to a power series will always have the same nice form. In general, we will compute

$$\lim_{n\to\infty} \frac{|a_{n+1}||x|^{n+1}}{|a_n||x|^n} = \lim_{n\to\infty} |x|\frac{|a_{n+1}|}{|a_n|} = |x| \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L|x|,$$

assuming that $\lim |a_{n+1}|/|a_n|$ exists. Then the series converges if $L|x| < 1$, that is, if $|x| < 1/L$, and diverges if $|x| > 1/L$. Only the two values $x = \pm 1/L$ require further investigation. Thus the series will always define a function on the interval $(-1/L, 1/L)$, that perhaps will extend to one or both endpoints as well. Two special cases deserve mention: if $L = 0$ the limit is 0 no matter what value x takes, so the series converges for all x and the function is defined for all real numbers. If $L = \infty$, then no matter what value x takes the limit is infinite and the series converges only when $x = 0$. The value $1/L$ is called the **radius of convergence** of the series, and the interval on which the series converges is the **interval of convergence**.

We can make these ideas a bit more general. Consider the series

$$\sum_{n=0}^{\infty} \frac{(x+2)^n}{3^n}$$

This looks a lot like a power series, but with $(x+2)^n$ instead of x^n. Let's try to determine the values of x for which it converges. This is just a geometric series, so it converges when

$$\begin{aligned} |x+2|/3 &< 1 \\ |x+2| &< 3 \\ -3 < x+2 &< 3 \\ -5 < x &< 1. \end{aligned}$$

So the interval of convergence for this series is $(-5, 1)$. The center of this interval is at -2, which is at distance 3 from the endpoints, so the radius of convergence is 3, and we say that the series is centered at -2.

Interestingly, if we compute the sum of the series we get

$$\sum_{n=0}^{\infty} \left(\frac{x+2}{3}\right)^n = \frac{1}{1 - \frac{x+2}{3}} = \frac{3}{1-x}.$$

Multiplying both sides by 1/3 we obtain

$$\sum_{n=0}^{\infty} \frac{(x+2)^n}{3^{n+1}} = \frac{1}{1-x},$$

which we recognize as being equal to

$$\sum_{n=0}^{\infty} x^n,$$

so we have two series with the same sum but different intervals of convergence.

This leads to the following definition:

Definition 6.166: Power Series

A power series centered at c has the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n,$$

where c and each a_n are real numbers.

Exercises for 6.8

Exercise 6.8.1 *Find the radius and interval of convergence for each series. In part c), do not attempt to determine whether the endpoints are in the interval of convergence.*

(a) $\sum_{n=0}^{\infty} nx^n$

(b) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

(c) $\sum_{n=1}^{\infty} \frac{n!}{n^n}(x-2)^n$

(d) $\sum_{n=1}^{\infty} \frac{(n!)^2}{n^n}(x-2)^n$

(e) $\sum_{n=1}^{\infty} \frac{(x+5)^n}{n(n+1)}$

Exercise 6.8.2 *Find the radius of convergence for the series* $\sum_{n=1}^{\infty} \frac{n!}{n^n}x^n$.

6.9 Calculus with Power Series

We now know that some functions can be expressed as power series, which look like infinite polynomials. Since it is easy to find derivatives and integrals of polynomials, we might hope that we can take derivatives and integrals of power series in an analogous way. In fact we can, as stated in the following theorem, which we will not prove here.

Theorem 6.167:

Suppose the power series $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ *has radius of convergence* R. *Then*

$$f'(x) = \sum_{n=0}^{\infty} na_n(x-a)^{n-1},$$
$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-a)^{n+1},$$

and these two series have radius of convergence R.

Example 6.168:

Find a power series representation of $\ln|1-x|$.

Solution. Starting with the geometric series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\int \frac{1}{1-x}\,dx = -\ln|1-x| = \sum_{n=0}^{\infty} \frac{1}{n+1}x^{n+1}$$
$$\ln|1-x| = \sum_{n=0}^{\infty} -\frac{1}{n+1}x^{n+1}$$

when $|x| < 1$. The series does not converge when $x = 1$ but does converge when $x = -1$ or $1 - x = 2$. The interval of convergence is $[-1,1)$, or $0 < 1 - x \leq 2$. We can use this series to express $\ln(a)$ as a series when $0 < a \leq 2$ by setting $x - 1 = a$. For example

$$\ln(3/2) = \ln(1-(-1/2)) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1}\frac{1}{2^{n+1}}.$$

We can use this in turn to approximate $\ln(3/2)$:

$$\ln(3/2) \approx \frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \frac{1}{64} + \frac{1}{160} - \frac{1}{384} + \frac{1}{896} = \frac{909}{2240} \approx 0.406.$$

Because this is an alternating series with decreasing terms, we know that the true value is between $909/2240$ and $909/2240 - 1/2048 = 29053/71680 \approx .4053$, so $0.4053 \leq \ln(3/2) \leq 0.406$. ♣

With a bit of arithmetic, we can approximate values outside of the interval of convergence:

Example 6.169:

Find an approximation for $\ln(9/4)$.

Solution. We can use the approximation we just computed, plus some rules for logarithms:

$$\ln(9/4) = \ln((3/2)^2) = 2\ln(3/2) \approx 0.812,$$

and using our bounds above,

$$0.8106 \leq \ln(9/4) \leq 0.812.$$

♣

Exercises for 6.9

Exercise 6.9.1 *Find a series representation for* $\ln 2$.

Exercise 6.9.2 *Find a power series representation for* $1/(1-x)^2$.

Exercise 6.9.3 *Find a power series representation for* $2/(1-x)^3$.

Exercise 6.9.4 *Find a power series representation for* $1/(1-x)^3$. *What is the radius of convergence?*

Exercise 6.9.5 *Find a power series representation for* $\int \ln(1-x)\,dx$.

6.10 Taylor Series

We have seen that some functions can be represented as series, which may give valuable information about the function. So far, we have seen only those examples that result from manipulation of our one fundamental example, the geometric series. We would like to start with a given function and produce a series to represent it, if possible.

Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ on some interval of convergence centered at 0. Then we know that we can compute derivatives of f by taking derivatives of the terms of the series. Let's look at the first few in general:

$$
\begin{aligned}
f'(x) &= \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots \\
f''(x) &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + \cdots \\
f'''(x) &= \sum_{n=3}^{\infty} n(n-1)(n-2) a_n x^{n-3} = 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4 x + \cdots
\end{aligned}
$$

By examining these it's not hard to discern the general pattern. The kth derivative must be

$$
\begin{aligned}
f^{(k)}(x) &= \sum_{n=k}^{\infty} n(n-1)(n-2)\cdots(n-k+1) a_n x^{n-k} \\
&= k(k-1)(k-2)\cdots(2)(1)a_k + (k+1)(k)\cdots(2)a_{k+1}x + \\
&\quad + (k+2)(k+1)\cdots(3)a_{k+2}x^2 + \cdots
\end{aligned}
$$

We can express this more clearly by using factorial notation:

$$
f^{(k)}(x) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n x^{n-k} = k!a_k + (k+1)!a_{k+1}x + \frac{(k+2)!}{2!} a_{k+2} x^2 + \cdots
$$

We can solve for a_n by substituting $x = 0$ in the formula for $f^{(k)}(x)$:

$$
f^{(k)}(0) = k!a_k + \sum_{n=k+1}^{\infty} \frac{n!}{(n-k)!} a_n 0^{n-k} = k!a_k,
$$

$$
a_k = \frac{f^{(k)}(0)}{k!}.
$$

Note that the original series for f yields $f(0) = a_0$.

So if a function f can be represented by a series, we can easily find such a series. Given a function f, the series

$$
\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n
$$

is called the **Maclaurin series** for f.

Example 6.170: Maclaurin Series

Find the Maclaurin series for $f(x) = 1/(1-x)$.

Solution. We need to compute the derivatives of f (and hope to spot a pattern).

$$\begin{aligned} f(x) &= (1-x)^{-1} \\ f'(x) &= (1-x)^{-2} \\ f''(x) &= 2(1-x)^{-3} \\ f'''(x) &= 6(1-x)^{-4} \\ f^{(4)}(x) &= 4!(1-x)^{-5} \\ &\vdots \\ f^{(n)}(x) &= n!(1-x)^{-n-1} \end{aligned}$$

So

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{n!(1-0)^{-n-1}}{n!} = 1$$

and the Maclaurin series is

$$\sum_{n=0}^{\infty} 1 \cdot x^n = \sum_{n=0}^{\infty} x^n,$$

the geometric series. ♣

A warning is in order here. Given a function f we may be able to compute the Maclaurin series, but that does not mean we have found a series representation for f. We still need to know where the series converges, and if, where it converges, it converges to $f(x)$. While for most commonly encountered functions the Maclaurin series does indeed converge to f on some interval, this is not true of all functions, so care is required.

As a practical matter, if we are interested in using a series to approximate a function, we will need some finite number of terms of the series. Even for functions with messy derivatives we can compute these using computer software like Sage. If we want to describe a series completely, we would like to be able to write down a formula for a typical term in the series. Fortunately, a few of the most important functions are very easy.

Example 6.171: Maclaurin Series

Find the Maclaurin series for $\sin x$.

Solution. Computing the first few derivatives is simple: $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, $f^{(4)}(x) = \sin x$, and then the pattern repeats. The values of the derivative when $x = 0$ are: 1, 0, -1, 0, 1, 0, -1, 0,…, and so the Maclaurin series is

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

We should always determine the radius of convergence:

$$\lim_{n\to\infty} \frac{|x|^{2n+3}}{(2n+3)!} \frac{(2n+1)!}{|x|^{2n+1}} = \lim_{n\to\infty} \frac{|x|^2}{(2n+3)(2n+2)} = 0,$$

so the series converges for every x. Since it turns out that this series does indeed converge to $\sin x$ everywhere, we have a series representation for $\sin x$ for every x. ♣

Sometimes the formula for the nth derivative of a function f is difficult to discover, but a combination of a known Maclaurin series and some algebraic manipulation leads easily to the Maclaurin series for f.

Example 6.172: Maclaurin Series

Find the Maclaurin series for $x\sin(-x)$.

Solution. To get from $\sin x$ to $x\sin(-x)$ we substitute $-x$ for x and then multiply by x. We can do the same thing to the series for $\sin x$:

$$x\sum_{n=0}^{\infty}(-1)^n\frac{(-x)^{2n+1}}{(2n+1)!} = x\sum_{n=0}^{\infty}(-1)^n(-1)^{2n+1}\frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n+2}}{(2n+1)!}.$$

♣

As we have seen, a power series can be centered at a point other than zero, and the method that produces the Maclaurin series can also produce such series.

Example 6.173: Taylor Series

Find a series centered at -2 *for* $1/(1-x)$.

Solution. If the series is $\sum_{n=0}^{\infty} a_n(x+2)^n$ then looking at the kth derivative:

$$k!(1-x)^{-k-1} = \sum_{n=k}^{\infty}\frac{n!}{(n-k)!}a_n(x+2)^{n-k}$$

and substituting $x=-2$ we get $k!3^{-k-1} = k!a_k$ and $a_k = 3^{-k-1} = 1/3^{k+1}$, so the series is

$$\sum_{n=0}^{\infty}\frac{(x+2)^n}{3^{n+1}}.$$

♣

Such a series is called the **Taylor series** for the function, and the general term has the form

$$\frac{f^{(n)}(a)}{n!}(x-a)^n.$$

A Maclaurin series is simply a Taylor series with $a=0$.

Exercises for 6.10

Exercise 6.10.1 *For each function, find the Maclaurin series or Taylor series centered at a, and the radius of convergence.*

(a) $\cos x$

(b) e^x

(c) $1/x$, $a = 5$

(d) $\ln x$, $a = 1$

(e) $\ln x$, $a = 2$

(f) $1/x^2$, $a = 1$

(g) $1/\sqrt{1-x}$

(h) *Find the first four terms of the Maclaurin series for* $\tan x$ *(up to and including the* x^3 *term).*

(i) *Use a combination of Maclaurin series and algebraic manipulation to find a series centered at zero for* $x\cos(x^2)$.

(j) *Use a combination of Maclaurin series and algebraic manipulation to find a series centered at zero for* xe^{-x}.

6.11 Taylor's Theorem

One of the most important uses of infinite series is using an initial portion of the series for f to approximate f. We have seen, for example, that when we add up the first n terms of an alternating series with decreasing terms that the difference between this and the true value is at most the size of the next term. A similar result is true of many Taylor series.

Theorem 6.174:

Suppose that f is defined on some open interval I around a and suppose $f^{(N+1)}(x)$ exists on this interval. Then for each $x \neq a$ in I there is a value z between x and a so that

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(N+1)}(z)}{(N+1)!}(x-a)^{N+1}.$$

Proof. The proof requires some cleverness to set up, but then the details are quite elementary. We define a function $F(t)$ as follows:

$$F(t) = \sum_{n=0}^{N} \frac{f^{(n)}(t)}{n!}(x-t)^n + B(x-t)^{N+1}.$$

Here we have replaced a by t in the first $N+1$ terms of the Taylor series, and added a carefully chosen term on the end, with B to be determined. Note that we are temporarily keeping x fixed, so the only variable in this equation is t, and we will be interested only in t between a and x. Now substitute $t = a$:

$$F(a) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n + B(x-a)^{N+1}.$$

Set this equal to $f(x)$:

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n + B(x-a)^{N+1}.$$

Since $x \neq a$, we can solve this for B, which is a "constant"—it depends on x and a but those are temporarily fixed. Now we have defined a function $F(t)$ with the property that $F(a) = f(x)$. Also, all terms with a positive power of $(x-t)$ become zero when we substitute x for t, so $F(x) = f^{(0)}(x)/0! = f(x)$. So $F(a) = F(x)$. By Rolle's theorem, we know that there is a value $z \in (a,x)$ such that $F'(z) = 0$. But what is F'? Each term in $F(t)$, except the first term and the extra term involving B, is a product, so to take the derivative we use the product rule on each of these terms.

$$\begin{aligned} F(t) = f(t) &+ \frac{f^{(1)}(t)}{1!}(x-t)^1 + \frac{f^{(2)}(t)}{2!}(x-t)^2 + \frac{f^{(3)}(t)}{3!}(x-t)^3 + \cdots \\ &+ \frac{f^{(N)}(t)}{N!}(x-t)^N + B(x-t)^{N+1}. \end{aligned}$$

So the derivative is

$$\begin{aligned} F'(t) = f'(t) &+ \left(\frac{f^{(1)}(t)}{1!}(x-t)^0(-1) + \frac{f^{(2)}(t)}{1!}(x-t)^1\right) \\ &+ \left(\frac{f^{(2)}(t)}{1!}(x-t)^1(-1) + \frac{f^{(3)}(t)}{2!}(x-t)^2\right) \\ &+ \left(\frac{f^{(3)}(t)}{2!}(x-t)^2(-1) + \frac{f^{(4)}(t)}{3!}(x-t)^3\right) + \cdots + \\ &+ \left(\frac{f^{(N)}(t)}{(N-1)!}(x-t)^{N-1}(-1) + \frac{f^{(N+1)}(t)}{N!}(x-t)^N\right) \\ &+ B(N+1)(x-t)^N(-1). \end{aligned}$$

The second term in each parenthesis cancel with the first term in the next one, leaving just

$$F'(t) = \frac{f^{(N+1)}(t)}{N!}(x-t)^N + B(N+1)(x-t)^N(-1).$$

At some z, $F'(z) = 0$ so

$$\begin{aligned} 0 &= \frac{f^{(N+1)}(z)}{N!}(x-z)^N + B(N+1)(x-z)^N(-1) \\ B(N+1)(x-z)^N &= \frac{f^{(N+1)}(z)}{N!}(x-z)^N \\ B &= \frac{f^{(N+1)}(z)}{(N+1)!}. \end{aligned}$$

Now we can write

$$F(t) = \sum_{n=0}^{N} \frac{f^{(n)}(t)}{n!}(x-t)^n + \frac{f^{(N+1)}(z)}{(N+1)!}(x-t)^{N+1}.$$

Recalling that $F(a) = f(x)$ we get

$$f(x) = F(a) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(N+1)}(z)}{(N+1)!}(x-a)^{N+1},$$

which is what we wanted to show. ♣

It may not be immediately obvious that this is particularly useful; let's look at some examples.

Example 6.175:

Find a polynomial approximation for $\sin x$ *accurate to* ± 0.005 *for values of* x *in* $[-\pi/2, \pi/2]$.

Solution. From Taylor's theorem with $a = 0$:

$$\sin x = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!} x^n + \frac{f^{(N+1)}(z)}{(N+1)!} x^{N+1}.$$

What can we say about the size of the term

$$\frac{f^{(N+1)}(z)}{(N+1)!} x^{N+1}?$$

Every derivative of $\sin x$ is $\pm \sin x$ or $\pm \cos x$, so $|f^{(N+1)}(z)| \leq 1$.

So we need to pick N so that

$$\left| \frac{x^{N+1}}{(N+1)!} \right| < 0.005.$$

Since we have limited x to $[-\pi/2, \pi/2]$,

$$\left| \frac{x^{N+1}}{(N+1)!} \right| < \frac{2^{N+1}}{(N+1)!}.$$

The quantity on the right decreases with increasing N, so all we need to do is find an N so that

$$\frac{2^{N+1}}{(N+1)!} < 0.005.$$

A little trial and error shows that $N = 8$ works, and in fact $2^9/9! < 0.0015$, so

$$\sin x = \sum_{n=0}^{8} \frac{f^{(n)}(0)}{n!} x^n \pm 0.0015$$

$$= x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} \pm 0.0015.$$

Figure 6.43 shows the graphs of $\sin x$ and and the approximation on $[0, 3\pi/2]$. As x gets larger, the approximation heads to negative infinity very quickly, since it is essentially acting like $-x^7$. ♣

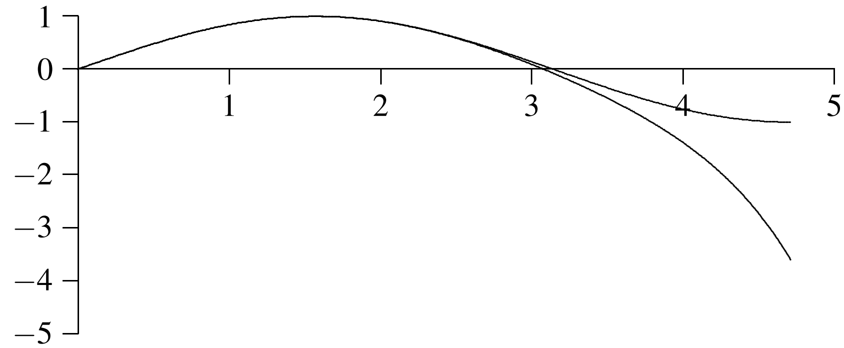


Figure 6.43: $\sin x$ **and a polynomial approximation.**

Note that we can now approximate the value of $\sin(x)$ to within 0.005 by using simple trigonometric identities to translate x into the interval $[-\pi/2, \pi/2]$.

We can extract a bit more information from this example. If we do not limit the value of x, we still have

$$\left|\frac{f^{(N+1)}(z)}{(N+1)!}x^{N+1}\right| \leq \left|\frac{x^{N+1}}{(N+1)!}\right|$$

so that $\sin x$ is represented by

$$\sum_{n=0}^{N}\frac{f^{(n)}(0)}{n!}x^n \pm \left|\frac{x^{N+1}}{(N+1)!}\right|.$$

If we can show that

$$\lim_{N\to\infty}\left|\frac{x^{N+1}}{(N+1)!}\right| = 0$$

for each x then

$$\sin x = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!},$$

that is, the sine function is actually equal to its Maclaurin series for all x. How can we prove that the limit is zero? Suppose that N is larger than $|x|$, and let M be the largest integer less than $|x|$ (if $M = 0$ the following is even easier). Then

$$\begin{aligned}\frac{|x^{N+1}|}{(N+1)!} &= \frac{|x|}{N+1}\frac{|x|}{N}\frac{|x|}{N-1}\cdots\frac{|x|}{M+1}\frac{|x|}{M}\frac{|x|}{M-1}\cdots\frac{|x|}{2}\frac{|x|}{1}\\ &\leq \frac{|x|}{N+1}\cdot 1\cdot 1\cdots 1\cdot\frac{|x|}{M}\frac{|x|}{M-1}\cdots\frac{|x|}{2}\frac{|x|}{1}\\ &= \frac{|x|}{N+1}\frac{|x|^M}{M!}.\end{aligned}$$

The quantity $|x|^M/M!$ is a constant, so

$$\lim_{N\to\infty}\frac{|x|}{N+1}\frac{|x|^M}{M!} = 0$$

and by the Squeeze Theorem (6.124)

$$\lim_{N\to\infty}\left|\frac{x^{N+1}}{(N+1)!}\right| = 0$$

as desired. Essentially the same argument works for $\cos x$ and e^x; unfortunately, it is more difficult to show that most functions are equal to their Maclaurin series.

Example 6.176:

Find a polynomial approximation for e^x near $x = 2$ accurate to ± 0.005.

Solution. From Taylor's theorem:

$$e^x = \sum_{n=0}^{N} \frac{e^2}{n!}(x-2)^n + \frac{e^z}{(N+1)!}(x-2)^{N+1},$$

since $f^{(n)}(x) = e^x$ for all n. We are interested in x near 2, and we need to keep $|(x-2)^{N+1}|$ in check, so we may as well specify that $|x-2| \le 1$, so $x \in [1,3]$. Also

$$\left|\frac{e^z}{(N+1)!}\right| \le \frac{e^3}{(N+1)!},$$

so we need to find an N that makes $e^3/(N+1)! \le 0.005$. This time $N = 5$ makes $e^3/(N+1)! < 0.0015$, so the approximating polynomial is

$$e^x = e^2 + e^2(x-2) + \frac{e^2}{2}(x-2)^2 + \frac{e^2}{6}(x-2)^3 + \frac{e^2}{24}(x-2)^4 + \frac{e^2}{120}(x-2)^5 \pm 0.0015.$$

Note that our approximation requires that we already have a very accurate approximation of the value e^2, which we shouldn't assume we have in the context of trying to approximate e^x. For this reason we typically try to center our series on values for which the derivative of the function is easy to evaluate (e.g. $a = 0$). ♣

Note well that in these examples we found polynomials of a certain accuracy only on a small interval, even though the series for $\sin x$ and e^x converge for all x; this is typical. To get the same accuracy on a larger interval would require more terms.

Exercises for 6.11

Exercise 6.11.1 *Find a polynomial approximation for $\cos x$ on $[0,\pi]$, accurate to $\pm 10^{-3}$*

Exercise 6.11.2 *How many terms of the series for $\ln x$ centered at 1 are required so that the guaranteed error on $[1/2, 3/2]$ is at most 10^{-3}? What if the interval is instead $[1, 3/2]$?*

Exercise 6.11.3 *Find the first three nonzero terms in the Taylor series for $\tan x$ on $[-\pi/4, \pi/4]$, and compute the guaranteed error term as given by Taylor's theorem. (You may want to use Sage or a similar aid.)*

Exercise 6.11.4 *Show that $\cos x$ is equal to its Taylor series for all x by showing that the limit of the error term is zero as N approaches infinity.*

Exercise 6.11.5 *Show that e^x is equal to its Taylor series for all x by showing that the limit of the error term is zero as N approaches infinity.*

Additional Material

Table of Integrals

Basic Integrals

1. $\int u^n du = \frac{u^{n+1}}{n+1} + C, n \neq -1$
2. $\int \frac{du}{u} = \ln|u| + C$
3. $\int e^u \, du = e^u + C$
4. $\int a^u \, du = \frac{a^u}{\ln a} + C$
5. $\int \sin u \, du = -\cos u + C$
6. $\int \cos u \, du = \sin u + C$
7. $\int \sec^2 u \, du = \tan u + C$
8. $\int \csc^2 u \, du = -\cot u + C$
9. $\int \sec u \tan u \, du = \sec u + C$
10. $\int \csc u \cot u \, du = -\csc u + C$
11. $\int \tan u \, du = \ln|\sec u| + C$
12. $\int \cot u \, du = \ln|\sin u| + C$
13. $\int \sec u \, du = \ln|\sec u + \tan u| + C$
14. $\int \csc u \, du = \ln|\csc u - \cot u| + C$
15. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a} + C$
16. $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C$
17. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{u}{a} + C$

Trigonometric Integrals

18. $\int \sin^2 u \, du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$
19. $\int \cos^2 u \, du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$
20. $\int \tan^2 u \, du = \tan u - u + C$
21. $\int \cot^2 u \, du = -\cot u - u + C$

22. $\int \sin^3 u\, du = -\frac{1}{3}(2+\sin^2 u)\cos u + C$

23. $\int \cos^3 u\, du = \frac{1}{3}(2+\cos^2 u)\sin u + C$

24. $\int \tan^3 u\, du = \frac{1}{2}\tan^2 u + \ln|\cos u| + C$

25. $\int \cot^3 u\, du = -\frac{1}{2}\cot^2 u - \ln|\sin u| + C$

26. $\int \sec^3 u\, du = \frac{1}{2}\sec u \tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$

27. $\int \csc^3 u\, du = -\frac{1}{2}\csc u \cot u + \frac{1}{2}\ln|\csc u - \cot u| + C$

28. $\int \sin^n u\, du = -\frac{1}{n}\sin^{n-1} u \cos u + \frac{n-1}{n}\int \sin^{n-2} u\, du$

29. $\int \cos^n u\, du = \frac{1}{n}\cos^{n-1} u \sin u + \frac{n-1}{n}\int \cos^{n-2} u\, du$

30. $\int \tan^n u\, du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u\, du$

31. $\int \cot^n u\, du = \frac{-1}{n-1}\cot^{n-1} u - \int \cot^{n-2} u\, du$

32. $\int \sec^n u\, du = \frac{1}{n-1}\tan u \sec^{n-2} u + \frac{n-2}{n-1}\int \sec^{n-2} u\, du$

33. $\int \csc^n u\, du = \frac{-1}{n-1}\cot u \csc^{n-2} u + \frac{n-2}{n-1}\int \csc^{n-2} u\, du$

34. $\int \sin au \sin bu\, du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C$

35. $\int \cos au \cos bu\, du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$

36. $\int \sin au \cos bu\, du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$

37. $\int u \sin u\, du = \sin u - u\cos u + C$

38. $\int u \cos u\, du = \cos u + u\sin u + C$

39. $\int u^n \sin u\, du = -u^n \cos u + n\int u^{n-1}\cos u\, du$

40. $\int u^n \cos u\, du = u^n \sin u - n\int u^{n-1}\sin u\, du$

41. $\int \sin^n u \cos^m u\, du = -\frac{\sin^{n-1} u \cos^{m+1} u}{n+m} + \frac{n-1}{n+m}\int \sin^{n-2} u \cos^m u\, du$
$= \frac{\sin^{n+1} u \cos^{m-1} u}{n+m} + \frac{m-1}{n+m}\int \sin^n u \cos^{m-2} u\, du$

Exponential and Logarithmic Integrals

42. $\int ue^{au}\,du = \frac{1}{a^2}(au-1)e^{au}+C$

43. $\int u^n e^{au}\,du = \frac{1}{a}u^n e^{au} - \frac{n}{a}\int u^{n-1}e^{au}\,du$

44. $\int e^{au}\sin bu\,du = \frac{e^{au}}{a^2+b^2}(a\sin bu - b\cos bu)+C$

45. $\int e^{au}\cos bu\,du = \frac{e^{au}}{a^2+b^2}(a\cos bu + b\sin bu)+C$

46. $\int \ln u\,du = u\ln u - u + C$

47. $\int u^n \ln u\,du = \frac{u^{n+1}}{(n+1)^2}\left[(n+1)\ln u - 1\right]+C$

48. $\int \frac{1}{u\ln u}\,du = \ln|\ln u| + C$

Hyperbolic Integrals

49. $\int \sinh u\,du = \cosh u + C$

50. $\int \cosh u\,du = \sinh u + C$

51. $\int \tanh u\,du = \ln\cosh u + C$

52. $\int \coth u\,du = \ln|\sinh u| + C$

53. $\int \text{sech} u\,du = tan^{-1}|\sinh u| + C$

54. $\int \text{csch} u\,du = \ln|\tanh \frac{1}{2}u| + C$

55. $\int \text{sech}^2 u\,du = \tanh u + C$

56. $\int \text{csch}^2 u\,du = -\coth u + C$

57. $\int \text{sech} u \tanh u\,du = -\text{sech} u + C$

58. $\int \text{csch} u \coth u\,du = -\text{csch} u + C$

Inverse Trigonometric Integrals

59. $\int \sin^{-1}u\,du = u\sin^{-1}u + \sqrt{1-u^2}+C$

60. $\int \cos^{-1}u\,du = u\cos^{-1}u - \sqrt{1-u^2}+C$

61. $\int \tan^{-1}u\,du = u\tan^{-1}u - \frac{1}{2}\ln(1+u^2)+C$

62. $\int u\sin^{-1}u\,du = \frac{2u^2-1}{4}\sin^{-1}u + \frac{u\sqrt{1-u^2}}{4}+C$

63. $\int u\cos^{-1}u\,du = \frac{2u^2-1}{4}\cos^{-1}u - \frac{u\sqrt{1-u^2}}{4} + C$

64. $\int u\tan^{-1}u\,du = \frac{u^2+1}{2}\tan^{-1}u - \frac{u}{2} + C$

65. $\int u^n\sin^{-1}u\,du = \frac{1}{n+1}\left[u^{n+1}\sin^{-1}u - \int\frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], n \neq -1$

66. $\int u^n\cos^{-1}u\,du = \frac{1}{n+1}\left[u^{n+1}\cos^{-1}u + \int\frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], n \neq -1$

67. $\int u^n\tan^{-1}u\,du = \frac{1}{n+1}\left[u^{n+1}\tan^{-1}u - \int\frac{u^{n+1}\,du}{1+u^2}\right], n \neq -1$

Integrals Involving a^2+u^2, $a>0$

68. $\int\sqrt{a^2+u^2}\,du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln(u+\sqrt{a^2+u^2}) + C$

69. $\int u^2\sqrt{a^2+u^2}\,du = \frac{u}{8}(a^2+2u^2)\sqrt{a^2+u^2} - \frac{a^4}{8}\ln(u+\sqrt{a^2+u^2}) + C$

70. $\int\frac{\sqrt{a^2+u^2}}{u}\,du = \sqrt{a^2+u^2} - a\ln\left|\frac{a+\sqrt{a^2+u^2}}{u}\right| + C$

71. $\int\frac{\sqrt{a^2+u^2}}{u^2}\,du = -\frac{\sqrt{a^2+u^2}}{u} + \ln(u+\sqrt{a^2+u^2}) + C$

72. $\int\frac{du}{\sqrt{a^2+u^2}} = \ln(u+\sqrt{a^2+u^2}) + C$

73. $\int\frac{u^2\,du}{\sqrt{a^2+u^2}} = \frac{u}{2}(\sqrt{a^2+u^2}) - \frac{a^2}{2}\ln(u+\sqrt{a^2+u^2}) + C$

74. $\int\frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\ln\left|\frac{\sqrt{a^2+u^2}+a}{u}\right| + C$

75. $\int\frac{du}{u^2\sqrt{a^2+u^2}} = -\frac{\sqrt{a^2+u^2}}{a^2u} + C$

76. $\int\frac{du}{(a^2+u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2+u^2}} + C$

Integrals Involving $u^2 - a^2$, $a > 0$

77. $\int \sqrt{u^2 - a^2}\, du = \frac{u}{2}\sqrt{u^2 - a^2} - \frac{a^2}{2} \ln\left|u + \sqrt{u^2 - a^2}\right| + C$

78. $\int u^2\sqrt{u^2 - a^2}\, du = \frac{u}{8}(2u^2 - a^2)\sqrt{u^2 - a^2} - \frac{a^4}{8} \ln\left|u + \sqrt{u^2 - a^2}\right| + C$

79. $\int \frac{\sqrt{u^2 - a^2}}{u}\, du = \sqrt{u^2 - a^2} - a\cos^{-1}\frac{a}{|u|} + C$

80. $\int \frac{\sqrt{u^2 - a^2}}{u^2}\, du = -\frac{\sqrt{u^2 - a^2}}{u} + \ln\left|u + \sqrt{u^2 - a^2}\right| + C$

81. $\int \frac{du}{\sqrt{u^2 - a^2}} = \ln|u + \sqrt{u^2 - a^2}| + C$

82. $\int \frac{u^2\, du}{\sqrt{u^2 - a^2}} = \frac{u}{2}(\sqrt{u^2 - a^2}) + \frac{a^2}{2} \ln\left|u + \sqrt{u^2 - a^2}\right| + C$

83. $\int \frac{du}{u^2\sqrt{u^2 - a^2}} = \frac{\sqrt{u^2 - a^2}}{a^2 u} + C$

84. $\int \frac{du}{(u^2 - a^2)^{3/2}} = -\frac{u}{a^2\sqrt{u^2 - a^2}} + C$

Integrals Involving $a^2 - u^2$, $a > 0$

85. $\int \sqrt{a^2 - u^2}\, du = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

86. $\int u^2\sqrt{a^2 - u^2}\, du = \frac{u}{8}(2u^2 - a^2)\sqrt{a^2 - u^2} + \frac{a^4}{8}\sin^{-1}\frac{u}{a} + C$

87. $\int \frac{\sqrt{a^2 - u^2}}{u}\, du = \sqrt{a^2 - u^2} - a\ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C$

88. $\int \frac{\sqrt{a^2 - u^2}}{u^2}\, du = -\frac{1}{u}\sqrt{a^2 - u^2} - \sin^{-1}\frac{u}{a} + C$

89. $\int \frac{u^2\, du}{\sqrt{a^2 - u^2}} = -\frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

90. $\int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a}\ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C$

91. $\int \frac{du}{u^2\sqrt{a^2 - u^2}} = -\frac{1}{a^2 u}\sqrt{a^2 - u^2} + C$

92. $\int (a^2 - u^2)^{3/2}\, du = -\frac{u}{8}(2u^2 - 5a^2)\sqrt{a^2 - u^2} + \frac{3a^4}{8}\sin^{-1}\frac{u}{a} + C$

93. $\int \frac{du}{(a^2 - u^2)^{3/2}} = -\frac{u}{a^2\sqrt{a^2 - u^2}} + C$

Integrals Involving $2au - u^2$, $a > 0$

94. $\int \sqrt{2au - u^2}\, du = \frac{u-a}{2}\sqrt{2au - u^2} + \frac{a^2}{2}\cos^{-1}(\frac{a-u}{a}) + C$

95. $\int \frac{du}{\sqrt{2au - u^2}} = \cos^{-1}(\frac{a-u}{a}) + C$

96. $\int u\sqrt{2au - u^2}\, du = \frac{2u^2 - au - 3a^2}{6}\sqrt{2au - u^2} + \frac{a^3}{2}\cos^{-1}(\frac{a-u}{a}) + C$

97. $\int \frac{du}{u\sqrt{2au - u^2}} = -\frac{\sqrt{2au - u^2}}{au} + C$

Integrals Involving $a + bu$, $a \neq 0$

98. $\int \frac{u\, du}{a + bu} = \frac{1}{b^2}(a + bu - a\ln|a + bu|) + C$

99. $\int \frac{u^2\, du}{a + bu} = \frac{1}{2b^3}\left[(a + bu)^2 - 4a(a + bu) + 2a^2\ln|a + bu|\right] + C$

100. $\int \frac{du}{u(a + bu)} = \frac{1}{a}\ln\left|\frac{u}{a + bu}\right| + C$

101. $\int \frac{du}{u^2(a + bu)} = -\frac{1}{au} + \frac{b}{a^2}\ln\left|\frac{a + bu}{u}\right| + C$

102. $\int \frac{u\, du}{(a + bu)^2} = \frac{a}{b^2(a + bu)} + \frac{1}{b^2}\ln|a + bu| + C$

103. $\int \frac{u\, du}{u(a + bu)^2} = \frac{1}{a(a + bu)} - \frac{1}{a^2}\ln\left|\frac{a + bu}{u}\right| + C$

104. $\int \frac{u^2\, du}{(a + bu)^2} = \frac{1}{b^3}(a + bu - \frac{a^2}{a + bu} - 2a\ln|a + bu|) + C$

105. $\int u\sqrt{a + bu}\, du = \frac{2}{15b^2}(3bu - 2a)(a + bu)^{3/2} + C$

106. $\int \frac{u\, du}{\sqrt{a + bu}} = \frac{2}{3b^2}(bu - 2a)\sqrt{a + bu} + C$

107. $\int \frac{u^2\, du}{\sqrt{a + bu}} = \frac{2}{15b^3}(8a^2 + 3b^2u^2 - 4abu)\sqrt{a + bu} + C$

108. $\int \frac{du}{u\sqrt{a + bu}} = \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a + bu} - \sqrt{a}}{\sqrt{a + bu} + \sqrt{a}}\right| + C$, if $a > 0$,
$= \frac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a + bu}{-a}} + C$, if $a < 0$

109. $\int \frac{\sqrt{a + bu}}{u}\, du = 2\sqrt{a + bu} + a\int \frac{du}{u\sqrt{a + bu}}$

110. $\int \frac{\sqrt{a+bu}}{u^2}\,du = -\frac{\sqrt{a+bu}}{u} + \frac{b}{2}\int \frac{du}{u\sqrt{a+bu}}$

111. $\int u^n\sqrt{a+bu}\,du = \frac{2}{b(2n+3)}\left[u^n(a+bu)^{3/2} - na\int u^{n-1}\sqrt{a+bu}\,du\right]$

112. $\int \frac{u^n\,du}{\sqrt{a+bu}} = \frac{2u^n\sqrt{a+bu}}{b(2n+1)} - \frac{2na}{b(2n+1)}\int \frac{u^{n-1}\,du}{\sqrt{a+bu}}$

113. $\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{du}{u^{n-1}\sqrt{a+bu}}$

Selected Exercise Answers

1.2.2 Compose $f(g(x))$ and $g(f(x))$ to confirm that each equals x.

1.2.3 Compose $f(g(x))$ and $g(f(x))$ to confirm that each equals x.

1.2.4 Compose $f(g(x))$ and $g(f(x))$ to confirm that each equals x.

1.2.5 Compose $f(g(x))$ and $g(f(x))$ to confirm that each equals x.

1.2.6 $\left(f^{-1}\right)'(20) = \frac{1}{f'(2)} = 1/5$

1.2.7 $\left(f^{-1}\right)'(7) = \frac{1}{f'(3)} = 1/4$

1.2.8 $\left(f^{-1}\right)'(\sqrt{3}/2) = \frac{1}{f'(\pi/6)} = 1$

1.2.9 $\left(f^{-1}\right)'(8) = \frac{1}{f'(1)} = 1/6$

1.2.10 $\left(f^{-1}\right)'(1/2) = \frac{1}{f'(1)} = -2$

1.2.11 $\left(f^{-1}\right)'(6) = \frac{1}{f'(0)} = 1/6$

1.2.12 $h'(t) = \frac{2}{\sqrt{1-4t^2}}$

1.2.13 $f'(t) = \frac{1}{|t|\sqrt{4t^2+1}}$

1.2.14 $g'(x) = \frac{2}{1+4x^2}$

1.2.15 $f'(x) = \frac{x}{\sqrt{1-x^2}} + \sin^{-1}(x)$

1.2.16 $g'(t) = \cos^{-1}(t)\cos(t) - \frac{\sin(t)}{\sqrt{1-t^2}}$

1.2.17 $f'(t) = \frac{e^t}{t} + \ln t e^t$

1.2.18 $h'(x) = \frac{\sin^{-1}(x)+\cos^{-1}(x)}{\sqrt{1-x^2}\cos^{-1}(x)^2}$

1.2.19 $g'(x) = \frac{1}{\sqrt{x}(2x+2)}$

1.2.20 $f'(x) = -\frac{1}{\sqrt{1-x^2}}$

1.2.21 (a) $f(x) = x$, so $f'(x) = 1$

(b) $f'(x) = \cos(\sin^{-1} x)\frac{1}{\sqrt{1-x^2}} = 1.$

1.3.1 $y = 2^x$

1.3.2 $y = 7$

1.3.3 $y = 2$

1.3.4 $x \neq 0$

1.5.1 $2\ln(3)x3^{x^2}$

1.5.2 $\dfrac{\cos x - \sin x}{e^x}$

1.5.3 $2e^{2x}$

1.5.4 $e^x \cos(e^x)$

1.5.5 $\cos(x)e^{\sin x}$

1.5.6 $x^{\sin x}\left(\cos x \ln x + \dfrac{\sin x}{x}\right)$

1.5.7 $3x^2e^x + x^3e^x$

1.5.8 $1 + 2^x \ln(2)$

1.5.9 $-2x\ln(3)(1/3)^{x^2}$

1.5.10 $e^{4x}(4x-1)/x^2$

1.5.11 $(3x^2+3)/(x^3+3x)$

1.5.12 $-\tan(x)$

1.5.13 $(1-\ln(x^2))/(x^2\sqrt{\ln(x^2)})$

1.5.14 $\sec(x)$

1.5.15 $x^{\cos(x)}(\cos(x)/x - \cos(x)\ln(x))$

1.5.19 e

2.1.1 (a) $\pi/3$

(b) $3\pi/4$

2.1.2

(a) $\pi/4$

(b) $-\pi/3$

(c) $1/3$

(d) $-3/4$

2.1.3 $\sqrt{1-x^2}/x$ with domain $[-1,0)\cup(0,1]$.

2.2.1 Because $\cosh x$ is always positive.

2.2.2 The points on the left hand side can be defined as $(-\cosh x, \sinh x)$.

2.2.3

$$\begin{aligned}\coth^2 x - \operatorname{csch}^2 x &= \left(\frac{e^x+e^{-x}}{e^x-e^{-x}}\right)^2 - \left(\frac{2}{e^x-e^{-x}}\right)^2\\ &= \frac{(e^{2x}+2+e^{-2x})-(4)}{e^{2x}-2+e^{-2x}}\\ &= \frac{e^{2x}-2+e^{-2x}}{e^{2x}-2+e^{-2x}}\\ &= 1\end{aligned}$$

2.2.4

$$\begin{aligned}\cosh^2 x + \sinh^2 x &= \left(\frac{e^x+e^{-x}}{2}\right)^2 + \left(\frac{e^x-e^{-x}}{2}\right)^2\\ &= \frac{e^{2x}+2+e^{-2x}}{4} + \frac{e^{2x}-2+e^{-2x}}{4}\\ &= \frac{2e^{2x}+2e^{-2x}}{4}\\ &= \frac{e^{2x}+e^{-2x}}{2}\\ &= \cosh 2x.\end{aligned}$$

2.2.5

$$\begin{aligned}\cosh^2 x &= \left(\frac{e^x+e^{-x}}{2}\right)^2\\ &= \frac{e^{2x}+2+e^{-2x}}{4}\\ &= \frac{1}{2}\frac{(e^{2x}+e^{-2x})+2}{2}\\ &= \frac{1}{2}\left(\frac{e^{2x}+e^{-2x}}{2}+1\right)\\ &= \frac{\cosh 2x+1}{2}.\end{aligned}$$

2.2.6

$$\begin{aligned}\sinh^2 x &= \left(\frac{e^x-e^{-x}}{2}\right)^2\\ &= \frac{e^{2x}-2+e^{-2x}}{4}\\ &= \frac{1}{2}\frac{(e^{2x}+e^{-2x})-2}{2}\\ &= \frac{1}{2}\left(\frac{e^{2x}+e^{-2x}}{2}-1\right)\\ &= \frac{\cosh 2x-1}{2}.\end{aligned}$$

2.2.7

$$\begin{aligned}\frac{d}{dx}[\text{sech}x] &= \frac{d}{dx}\left[\frac{2}{e^x+e^{-x}}\right]\\ &= \frac{-2(e^x-e^{-x})}{(e^x+e^{-x})^2}\\ &= -\frac{2(e^x-e^{-x})}{(e^x+e^{-x})(e^x+e^{-x})}\\ &= -\frac{2}{e^x+e^{-x}}\cdot\frac{e^x-e^{-x}}{e^x+e^{-x}}\\ &= -\text{sech}x\tanh x\end{aligned}$$

2.2.8

$$\begin{aligned}\frac{d}{dx}[\coth x] &= \frac{d}{dx}\left[\frac{e^x+e^{-x}}{e^x-e^{-x}}\right]\\ &= \frac{(e^x-e^{-x})(e^x-e^{-x})-(e^x+e^{-x})(e^x+e^{-x})}{(e^x-e^{-x})^2}\\ &= \frac{e^{2x}+e^{-2x}-2-(e^{2x}+e^{-2x}+2)}{(e^x-e^{-x})^2}\\ &= -\frac{4}{(e^x-e^{-x})^2}\\ &= -\text{csch}^2 x\end{aligned}$$

2.2.9

$$\int \tanh x\, dx = \int \frac{\sinh x}{\cosh x}\, dx$$

Let $u = \cosh x$; $du = (\sinh x)dx$

$$\begin{aligned}&= \int \frac{1}{u}\, du\\ &= \ln|u| + C\\ &= \ln(\cosh x) + C.\end{aligned}$$

2.2.10

$$\int \coth x\, dx = \int \frac{\cosh x}{\sinh x}\, dx$$

Let $u = \sinh x$; $du = (\cosh x)dx$

$$\begin{aligned}&= \int \frac{1}{u}\, du\\ &= \ln|u| + C\\ &= \ln|\sinh x| + C.\end{aligned}$$

2.2.11 $2\sinh 2x$

2.2.12 $2x\sec^2(x^2)$

2.2.13 $\coth x$

2.2.14 $\sinh^2 x + \cosh^2 x$

2.2.15 $x\cosh x$

2.2.16 $\frac{-2x}{(x^2)\sqrt{1-x^4}}$

2.2.17 $\frac{3}{\sqrt{9x^2+1}}$

2.2.18 $\frac{4x}{\sqrt{4x^4-1}}$

2.2.19 $\frac{1}{1-(x+5)^2}$

2.2.20 $-\csc x$

2.2.21 $\sec x$

2.2.22 $y = x$

2.2.23 $y = 3/4(x - \ln 2) + 5/4$

2.2.24 $y = -72/125(x - \ln 3) + 9/25$

2.2.25 $y = x$

2.2.26 $y = (x - \sqrt{2}) + \cosh^{-1}(\sqrt{2}) \approx (x - 1.414) + 0.881$

2.2.27 $1/2 \ln(\cosh(2x)) + C$

2.2.28 $1/3 \sinh(3x - 7) + C$

2.2.29 $1/2 \sinh^2 x + C$ or $1/2 \cosh^2 x + C$

2.2.30 $x \sinh(x) - \cosh(x) + C$

2.2.31 $x \cosh(x) - \sinh(x) + C$

2.2.32 $\begin{cases} \frac{1}{3}\tanh^{-1}\left(\frac{x}{3}\right) + C & x^2 < 9 \\ \frac{1}{3}\coth^{-1}\left(\frac{x}{3}\right) + C & 9 < x^2 \end{cases} = \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x-1| + C$

2.2.33 $\cosh^{-1}(x^2/2) + C = \ln(x^2 + \sqrt{x^4 - 4}) + C$

2.2.34 $2/3 \sinh^{-1} x^{3/2} + C = 2/3 \ln(x^{3/2} + \sqrt{x^3 + 1}) + C$

2.2.35 $\frac{1}{16}\tan^{-1}(x/2) + \frac{1}{32}\ln|x-2| + \frac{1}{32}\ln|x+2| + C$

2.2.36 $\ln x - \ln|x+1| + C$

2.2.37 $\tan^{-1}(e^x) + C$

2.2.38 $x\sinh^{-1}x - \sqrt{x^2+1} + C$

2.2.39 $x\tanh^{-1}x + 1/2\ln|x^2-1| + C$

2.2.40 $\tan^{-1}(\sinh x) + C$

2.2.41 0

2.2.42 $3/2$

2.2.43 2

2.3.1 (d)

2.3.2 $3/[5(x+3)]$

2.3.3 $6+h$

2.3.4 (a) $[2,3)\cup(3,\infty)$

(b) $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$

2.3.5 $\{x : x \neq 0\}$

2.3.6 $g(x) = (5x+26)/3$

2.3.7 (c)

2.3.8 $f^{-1}(x) = f(x) = \ln\left(\dfrac{e^x}{e^x-1}\right)$ and its domain is $(0,\infty)$.

2.3.9 (a) $2-\ln 3$

(b) 1,3

(c) $(e^2-1)^2$

(d) 3

2.3.10 $-\pi$

2.3.11 $2\pi/5$

2.3.12 1

2.4.1 0

2.4.2 ∞

2.4.3 0

2.4.4 0

2.4.5 $1/6$

2.4.6 $1/16$

2.4.7 $3/2$

2.4.8 $-1/4$

2.4.9 -3

2.4.10 $1/2$

2.4.11 0

2.4.12 -1

2.4.13 $-1/2$

2.4.14 5

2.4.15 1

2.4.16 1

2.4.17 2

2.4.18 1

2.4.19 0

2.4.20 $1/2$

2.4.21 2

2.4.22 0

2.4.23 $1/2$

2.4.24 $-1/2$

2.4.25 2

2.4.26 0

2.4.27 ∞

2.4.28 0

2.4.29 5

2.4.30 $-1/2$

3.1.1 $\cos x + x\sin x + C$

3.1.2 $x^2\sin x - 2\sin x + 2x\cos x + C$

3.1.3 $(x-1)e^x + C$

3.1.4 $(1/2)e^{x^2} + C$

3.1.5 $(x/2) - \sin(2x)/4 + C =$
$(x/2) - (\sin x\cos x)/2 + C$

3.1.6 $x\ln x - x + C$

3.1.7 $(x^2\arctan x + \arctan x - x)/2 + C$

3.1.8 $-x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C$

3.1.9 $x^3\sin x + 3x^2\cos x - 6x\sin x - 6\cos x + C$

3.1.10 $x^2/4 - (\cos^2 x)/4 - (x\sin x\cos x)/2 + C$

3.1.11 $x/4 - (x\cos^2 x)/2 + (\cos x\sin x)/4 + C$

3.1.12 $x\arctan(\sqrt{x}) + \arctan(\sqrt{x}) - \sqrt{x} + C$

3.1.13 $2\sin(\sqrt{x}) - 2\sqrt{x}\cos(\sqrt{x}) + C$

3.1.14 $\sec x\csc x - 2\cot x + C$

3.1.15 $\sin x - x\cos x + C$

3.1.16 $-e^{-x} - xe^{-x} + C$

3.1.17 $-x^2\cos x + 2x\sin x + 2\cos x + C$

3.1.18 $-x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C$

3.1.19 $1/2e^{x^2} + C$

3.1.20 $x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$

3.1.21 $-\frac{1}{2}xe^{-2x} - \frac{e^{-2x}}{4} + C$

3.1.22 $1/2e^x(\sin x - \cos x) + C$

3.1.23 $1/5e^{2x}(\sin x + 2\cos x) + C$

3.1.24 $1/13e^{2x}(2\sin(3x) - 3\cos(3x)) + C$

3.1.25 $1/10e^{5x}(\sin(5x) + \cos(5x)) + C$

3.1.26 $-1/2\cos^2 x + C$

3.1.27 $\sqrt{1-x^2} + x\sin^{-1}(x) + C$

3.1.28 $x\tan^{-1}(2x) - \frac{1}{4}\ln\left|4x^2+1\right| + C$

3.1.29 $\frac{1}{2}x^2\tan^{-1}(x) - \frac{x}{2} + \frac{1}{2}\tan^{-1}(x) + C$

3.1.30 $\sqrt{1-x^2} + x\sin^{-1}x + C$

3.1.31 $\frac{1}{2}x^2\ln|x| - \frac{x^2}{4} + C$

3.1.32 $-\frac{x^2}{4} + \frac{1}{2}x^2\ln|x| + 2x - 2x\ln|x| + C$

3.1.33 $-\frac{x^2}{4} + \frac{1}{2}x^2\ln|x-1| - \frac{x}{2} - \frac{1}{2}\ln|x-1| + C$

3.1.34 $\frac{1}{2}x^2\ln\left(x^2\right) - \frac{x^2}{2} + C$

3.1.35 $\frac{1}{3}x^3\ln|x| - \frac{x^3}{9} + C$

3.1.36 $2x + x(\ln|x|)^2 - 2x\ln|x| + C$

3.1.37 $2x + x(\ln|x+1|) + (\ln|x+1|)^2 - 2x\ln|x+1| - 2\ln|x+1| + 2 + C$

3.1.38 $x\tan(x) + \ln|\cos(x)| + C$

3.1.39 $\ln|\sin(x)| - x\cot(x) + C$

3.1.40 $\frac{2}{5}(x-2)^{5/2} + \frac{4}{3}(x-2)^{3/2} + C$

3.1.41 $x\sec x - \ln|\sec x + \tan x| + C$

3.1.42 $-x\csc x - \ln|\csc x + \cot x| + C$

3.1.43 $1/2x\big(\sin(\ln x) - \cos(\ln x)\big) + C$

3.1.44 $2\sin(\sqrt{x}) - 2\sqrt{x}\cos(\sqrt{x}) + C$

3.1.45 $\frac{1}{2}x\ln|x| - \frac{x}{2} + C$

3.1.46 $2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$

3.1.47 $1/2x^2 + C$

3.1.48 π

3.1.49 $-2/e$

3.1.50 0

3.1.51 $\frac{3\pi^2}{2} - 12$

3.1.52 $1/2$

3.1.53 $6 - 2e$

3.1.54 $\frac{3}{4e^2} - \frac{5}{4e^4}$

3.1.55 $\frac{1}{2} + \frac{e^\pi}{2}$

3.1.56 $1/5\,(e^\pi + e^{-\pi})$

3.2.1 $x/2 - \sin(2x)/4 + C$

3.2.2 $-\cos x + (\cos^3 x)/3 + C$

3.2.3 $3x/8 - (\sin 2x)/4 + (\sin 4x)/32 + C$

3.2.4 $(\cos^5 x)/5 - (\cos^3 x)/3 + C$

3.2.5 $\sin x - (\sin^3 x)/3 + C$

3.2.6 $(\sin^3 x)/3 - (\sin^5 x)/5 + C$

3.2.7 $-2(\cos x)^{5/2}/5 + C$

3.2.8 $\tan x - \cot x + C$

3.2.9 $(\sec^3 x)/3 - \sec x + C$

3.2.10 $-\cos x + \sin x + C$

3.2.11 $\dfrac{3}{2}\ln|\sec x+\tan x|+\tan x+\dfrac{1}{2}\sec x\tan x+C$

3.2.12 $\dfrac{\tan^5(x^2)}{10}+C$

3.2.13 $-\frac{1}{5}\cos^5(x)+C$

3.2.14 $\frac{1}{4}\sin^4(x)+C$

3.2.15 $\frac{1}{5}\cos^5 x-\frac{1}{3}\cos^3 x+C$

3.2.16 $\frac{1}{6}\cos^6 x-\frac{1}{4}\cos^4 x+C$

3.2.17 $\frac{1}{11}\sin^{11} x-\frac{2}{9}\sin^9 x+\frac{1}{7}\sin^7 x+C$

3.2.18 $-\frac{1}{9}\sin^9(x)+\frac{3\sin^7(x)}{7}-\frac{3\sin^5(x)}{5}+\frac{\sin^3(x)}{3}+C$

3.2.19 $\frac{x}{8}-\frac{1}{32}\sin(4x)+C$

3.2.20 $\frac{1}{2}\left(-\frac{1}{8}\cos(8x)-\frac{1}{2}\cos(2x)\right)+C$

3.2.21 $\frac{1}{2}\left(-\frac{1}{3}\cos(3x)+\cos(-x)\right)+C$

3.2.22 $\frac{1}{2}\left(\frac{1}{4}\sin(4x)-\frac{1}{10}\sin(10x)\right)+C$

3.2.23 $\frac{1}{2}\left(\frac{1}{\pi}\sin(\pi x)-\frac{1}{3\pi}\sin(3\pi x)\right)+C$

3.2.24 $\frac{1}{2}\left(\sin(x)+\frac{1}{3}\sin(3x)\right)+C$

3.2.25 $\frac{1}{\pi}\sin(\frac{\pi}{2}x)+\frac{1}{3\pi}\sin(\pi x)+C$

3.2.26 $\frac{\tan^5(x)}{5}+C$

3.2.27 $\frac{1}{5}\tan^5 x+\frac{1}{3}\tan^3 x+C$

3.2.28 $\frac{\tan^6(x)}{6}+\frac{\tan^4(x)}{4}+C$

3.2.29 $\frac{\tan^4(x)}{4}+C$

3.2.30 $\frac{\sec^5(x)}{5}-\frac{\sec^3(x)}{3}+C$

3.2.31 $\frac{\sec^9(x)}{9}-\frac{2\sec^7(x)}{7}+\frac{\sec^5(x)}{5}+C$

3.2.32 $\frac{1}{3}\tan^3 x-\tan x+x+C$

3.2.33 $\frac{1}{4}\tan x\sec^3 x+\frac{3}{8}(\sec x\tan x+\ln|\sec x+\tan x|)+C$

3.2.34 $\frac{1}{2}(\sec x\tan x-\ln|\sec x+\tan x|)+C$

3.2.35 $\frac{1}{4}\tan x\sec^3 x-\frac{1}{8}(\sec x\tan x+\ln|\sec x+\tan x|)+C$

3.3.1 $x\sqrt{x^2-1}/2-\ln|x+\sqrt{x^2-1}|/2+C$

3.3.2 $x\sqrt{9+4x^2}/2+(9/4)\ln|2x+\sqrt{9+4x^2}|+C$

3.3.3 $-(1-x^2)^{3/2}/3+C$

3.3.4 $\arcsin(x)/8-\sin(4\arcsin x)/32+C$

3.3.5 $\ln|x+\sqrt{1+x^2}|+C$

3.3.6 $(x+1)\sqrt{x^2+2x}/2-\ln|x+1+\sqrt{x^2+2x}|/2+C$

3.3.7 $-\arctan x-1/x+C$

3.3.8 $2\arcsin(x/2)-x\sqrt{4-x^2}/2+C$

3.3.9 $\arcsin(\sqrt{x})-\sqrt{x}\sqrt{1-x}+C$

3.3.10 $(2x^2+1)\sqrt{4x^2-1}/24+C$

3.3.11 $\frac{1}{2}\left(x\sqrt{x^2+1}+\ln|\sqrt{x^2+1}+x|\right)+C$

3.3.12 $2\left(\frac{x}{4}\sqrt{x^2+4}+\ln|\frac{\sqrt{x^2+4}}{2}+\frac{x}{2}|\right)+C$

3.3.13 $\frac{1}{2}\left(\sin^{-1}x+x\sqrt{1-x^2}\right)+C$

3.3.14 $\frac{1}{2}\left(9\sin^{-1}(x/3)+x\sqrt{9-x^2}\right)+C$

3.3.15 $\frac{1}{2}x\sqrt{x^2-1}-\frac{1}{2}\ln|x+\sqrt{x^2-1}|+C$

3.3.16 $\frac{1}{2}x\sqrt{x^2-16}-8\ln\left|\frac{x}{4}+\frac{\sqrt{x^2-16}}{4}\right|+C$

3.3.17 $x\sqrt{x^2+1/4}+\frac{1}{4}\ln|2\sqrt{x^2+1/4}+2x|+C=\frac{1}{2}x\sqrt{4x^2+1}+\frac{1}{4}\ln|\sqrt{4x^2+1}+2x|+C$

3.3.18 $\frac{1}{6}\sin^{-1}(3x)+\frac{3}{2}\sqrt{1/9-x^2}+C=\frac{1}{6}\sin^{-1}(3x)+\frac{1}{2}\sqrt{1-9x^2}+C$

3.3.19 $4\left(\frac{1}{2}x\sqrt{x^2-1/16}-\frac{1}{32}\ln|4x+4\sqrt{x^2-1/16}|\right)+C=\frac{1}{2}x\sqrt{16x^2-1}-\frac{1}{8}\ln|4x+\sqrt{16x^2-1}|+C$

3.3.20 $8\ln\left|\frac{\sqrt{x^2+2}}{\sqrt{2}}+\frac{x}{\sqrt{2}}\right|+C$

3.3.21 $3\sin^{-1}\left(\frac{x}{\sqrt{7}}\right)+C$ (Trig. Subst. is not needed)

3.3.22 $5\ln\left|\frac{x}{\sqrt{8}}+\frac{\sqrt{x^2-8}}{\sqrt{8}}\right|+C$

3.4.1 $-\ln|x-2|/4+\ln|x+2|/4+C$

3.4.2 $-x^3/3-4x-4\ln|x-2|+4\ln|x+2|+C$

3.4.3 $-1/(x+5)+C$

3.4.4 $-x-\ln|x-2|+\ln|x+2|+C$

3.4.5 $-4x+x^3/3+8\arctan(x/2)+C$

3.4.6 $(1/2)\arctan(x/2+5/2)+C$

3.4.7 $x^2/2-2\ln(4+x^2)+C$

3.4.8 $(1/4)\ln|x+3|-(1/4)\ln|x+7|+C$

3.4.9 $(1/5)\ln|2x-3|-(1/5)\ln|1+x|+C$

3.4.10 $(1/3)\ln|x|-(1/3)\ln|x+3|+C$

3.4.11 $3\ln|x-2|+4\ln|x+5|+C$

3.4.12 $9\ln|x+1|-2\ln|x|+C$

3.4.13 $\frac{1}{3}(\ln|x+2|-\ln|x-2|)+C$

3.4.14 $\ln|x+5|-\frac{2}{x+5}+C$

3.4.15 $-\frac{4}{x+8}-3\ln|x+8|+C$

3.4.16 $\frac{5}{x+1}+7\ln|x|+2\ln|x+1|+C$

3.4.17 $-\ln|2x-3|+5\ln|x-1|+2\ln|x+3|+C$

3.4.18 $-\frac{1}{5}\ln|5x-1|+\frac{2}{3}\ln|3x-1|+\frac{3}{7}\ln|7x+3|+C$

3.4.19 $x+\ln|x-1|-\ln|x+2|+C$

3.4.20 $\frac{x^2}{2}+x+\frac{125}{9}\ln|x-5|+\frac{64}{9}\ln|x+4|-\frac{35}{2}+C$

3.4.21 $2x+C$

3.4.22 $\frac{1}{6}\left(-\ln\left|x^2+2x+3\right|+2\ln|x|-\sqrt{2}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)\right)+C$

3.4.23 $-\frac{3}{2}\ln\left|x^2+4x+10\right|+x+\frac{\tan^{-1}\left(\frac{x+2}{\sqrt{6}}\right)}{\sqrt{6}}+C$

3.4.24 $\ln\left|3x^2+5x-1\right|+2\ln|x+1|+C$

3.4.25 $2\ln|x-3|+2\ln|x^2+6x+10|-4\tan^{-1}(x+3)+C$

3.4.26 $\frac{9}{10}\ln\left|x^2+9\right|+\frac{1}{5}\ln|x+1|-\frac{4}{15}\tan^{-1}\left(\frac{x}{3}\right)+C$

3.4.27 $\frac{1}{2}\left(3\ln\left|x^2+2x+17\right|-4\ln|x-7|+\tan^{-1}\left(\frac{x+1}{4}\right)\right)+C$

3.4.28 $3\left(\ln\left|x^2-2x+11\right|+\ln|x-9|\right)+3\sqrt{\frac{2}{5}}\tan^{-1}\left(\frac{x-1}{\sqrt{10}}\right)+C$

3.4.29 $\frac{1}{2}\ln\left|x^2+10x+27\right|+5\ln|x+2|-6\sqrt{2}\tan^{-1}\left(\frac{x+5}{\sqrt{2}}\right)+C$

3.4.30 $\ln(2000/243)\approx 2.108$

3.4.31 $5\ln(9/4)-\frac{1}{3}\ln(17/2)\approx 3.3413$

3.4.32 $-\pi/4+\tan^{-1}3-\ln(11/9)\approx 0.263$

3.4.33 $1/8$

3.5.1 $2^2+3^2+4^2=29$

3.5.2 $-6-2+2+6+10=10$

3.5.3 $0-1+0+1+0=0$

3.5.4 $1+1/2+1/3+1/4+1/5=137/60$

3.5.5 $-1+2-3+4-5+6=3$

3.5.6 $1/2+1/6+1/12+1/20=4/5$

3.5.7 $1+1+1+1+1+1=6$

3.5.8 Answers may vary; $\sum_{i=1}^{5}3i$

3.5.9 Answers may vary; $\sum_{i=0}^{8}(i^2-1)$

3.5.10 Answers may vary; $\sum_{i=1}^{4}\frac{i}{i+1}$

3.5.11 Answers may vary; $\sum_{i=0}^{4}(-1)^i e^i$

3.5.12 325

3.5.13 1045

3.5.14 $28,650$

3.5.15 -8525

3.5.16 2050

3.5.17 5050

3.5.18 2870

3.5.19 19

3.5.20 $59/8$

3.5.21 $\pi/3+\pi/(2\sqrt{3})\approx 1.954$

3.5.22 8.144

3.5.23 0.388584

3.5.24 $496/315\approx 1.5746$

3.5.25 (a) Exact expressions will vary; $\frac{(1+n)^2}{4n^2}$.

(b) $121/400$, $10201/40000$, $1002001/4000000$

(c) $1/4$

3.5.26 (a) Exact expressions will vary; $2+4/n^2$.

(b) $51/25$, $5001/2500$, $500001/250000$

(c) 2

3.5.27 (a) 8.

(b) 8, 8, 8

(c) 8

3.5.28 (a) Exact expressions will vary; $20/3-96/(3n)+64/(3n^2)$.

(b) $92/25$, $3968/625$, $103667/15625$

(c) $20/3$

3.5.29 (a) Exact expressions will vary; $100-200/n$.

(b) 80, 98, $499/5$

(c) 100

3.5.30 (a) Exact expressions will vary; $-1/12(1-1/n^2)$.

(b) $-33/400$, $-3333/40000$, $-333333/4000000$

(c) $-1/12$

3.6.1 T,S: 4 ± 0

3.6.2 T: 9.28125 ± 0.281125; S: 9 ± 0

3.6.3 T: 60.75 ± 1; S: 60 ± 0

3.6.4 T: 1.1167 ± 0.0833; S: 1.1000 ± 0.0167

3.6.5 T: 0.3235 ± 0.0026; S: 0.3217 ± 0.000065

3.6.6 T: 0.6478 ± 0.0052; S: 0.6438 ± 0.000033

3.6.7 T: 2.8833 ± 0.0834; S: 2.9000 ± 0.0167

3.6.8 T: 1.1170 ± 0.0077; S: 1.1114 ± 0.0002

3.6.9 T: 1.097 ± 0.0147; S: 1.089 ± 0.0003

3.6.10 T: 3.63 ± 0.087; S: 3.62 ± 0.032

3.7.1 Converges to 1.

3.7.2 Diverges.

3.7.3 1/3

3.7.4 Divergent.

3.7.7 (a) $\pi/2$

(b) divergent (to ∞)

(c) 1

(d) divergent (to ∞)

(e) $\frac{5}{3}(4^{3/5})$

3.7.9 $0 < p < 1$

3.7.11 $e^5/2$

3.7.12 $1/2$

3.7.13 $1/3$

3.7.14 $\pi/3$

3.7.15 $1/\ln 2$

3.7.16 diverges

3.7.17 diverges

3.7.18 $\pi/2$

3.7.19 1

3.7.20 diverges

3.7.21 diverges

3.7.22 diverges

3.7.23 diverges

3.7.24 diverges

3.7.25 diverges

3.7.26 $2+2\sqrt{2}$

3.7.27 1

3.7.28 $1/2$

3.7.29 0

3.7.30 $\pi/2$

3.7.31 $-1/4$

3.7.32 diverges

3.7.33 -1

3.7.34 1

3.7.35 diverges

3.7.36 $1/2$

3.7.37 $1/2$

3.8.1 $\dfrac{(t+4)^4}{4}+C$

3.8.2 $\dfrac{(t^2-9)^{5/2}}{5}+C$

3.8.3 $\dfrac{(e^{t^2}+16)^2}{4}+C$

3.8.4 $\cos t-\dfrac{2}{3}\cos^3 t+C$

3.8.5 $\dfrac{\tan^2 t}{2}+C$

3.8.6 $\ln|t^2+t+3|+C$

3.8.7 $\dfrac{1}{8}\ln|1-4/t^2|+C$

3.8.8 $\dfrac{1}{25}\tan(\arcsin(t/5))+C=\dfrac{t}{25\sqrt{25-t^2}}+C$

3.8.9 $\dfrac{2}{3}\sqrt{\sin 3t}+C$

3.8.10 $t\tan t+\ln|\cos t|+C$

3.8.11 $2\sqrt{e^t+1}+C$

3.8.12 $\dfrac{3t}{8}+\dfrac{\sin 2t}{4}+\dfrac{\sin 4t}{32}+C$

3.8.13 $\dfrac{\ln|t|}{3}-\dfrac{\ln|t+3|}{3}+C$

3.8.14 $\dfrac{-1}{\sin\arctan t}+C=-\sqrt{1+t^2}/t+C$

3.8.15 $\dfrac{-1}{2(1+\tan t)^2}+C$

3.8.16 $\dfrac{(t^2+1)^{5/2}}{5}-\dfrac{(t^2+1)^{3/2}}{3}+C$

3.8.17 $\dfrac{e^t\sin t-e^t\cos t}{2}+C$

3.8.18 $\dfrac{(t^{3/2}+47)^4}{6}+C$

3.8.19 $\dfrac{2}{3(2-t^2)^{3/2}}-\dfrac{1}{(2-t^2)^{1/2}}+C$

3.8.20 $\dfrac{\ln|\sin(\arctan(2t/3))|}{9}+C=(\ln(4t^2)-\ln(9+4t^2))/18+C$

3.8.21 $\dfrac{(\arctan(2t))^2}{4}+C$

3.8.22 $\dfrac{3\ln|t+3|}{4}+\dfrac{\ln|t-1|}{4}+C$

3.8.23 $\dfrac{\cos^7 t}{7}-\dfrac{\cos^5 t}{5}+C$

3.8.24 $\dfrac{-1}{t-3}+C$

3.8.25 $\dfrac{-1}{\ln t}+C$

3.8.26 $\dfrac{t^2(\ln t)^2}{2}-\dfrac{t^2\ln t}{2}+\dfrac{t^2}{4}+C$

3.8.27 $(t^3-3t^2+6t-6)e^t+C$

3.8.28 $\dfrac{5+\sqrt{5}}{10}\ln(2t+1-\sqrt{5})+\dfrac{5-\sqrt{5}}{10}\ln(2t+1+\sqrt{5})+C$

4.1.5 $8\pi/3$

4.1.6 $\pi/30$

4.1.7 $\pi(\pi/2-1)$

4.1.8

(a) $114\pi/5$

(b) $74\pi/5$

(c) 20π

(d) 4π

4.1.9 16π, 24π

4.1.11 $\pi h^2(3r-h)/3$

4.1.13 2π

4.2.1 $(22\sqrt{22}-8)/27$

4.2.2 $\ln(2)+3/8$

4.2.3 $a+a^3/3$

4.2.4 $\ln((\sqrt{2}+1)/\sqrt{3})$

4.2.6 $3/4$

4.2.7 ≈ 3.82

4.2.8 ≈ 1.01

4.2.9 $\sqrt{1+e^2}-\sqrt{2}+\frac{1}{2}\ln\left(\frac{\sqrt{1+e^2}-1}{\sqrt{1+e^2}+1}\right)+\frac{1}{2}\ln(3+2\sqrt{2})$

4.2.10 $\sqrt{2}$

4.2.11 6

4.2.12 $4/3$

4.2.13 6

4.2.14 $109/2$

4.2.15 $3/2$

4.2.16 $12/5$

4.2.17 $79953333/400000 \approx 199.883$

4.2.18 $-\ln(2-\sqrt{3}) \approx 1.31696$

4.2.19 $\sinh^{-1} 1$

4.2.20 $\int_0^1 \sqrt{1+4x^2}\,dx$

4.2.21 $\int_0^1 \sqrt{1+100x^{18}}\,dx$

4.2.22 $\int_0^1 \sqrt{1+\frac{1}{4x}}\,dx$

4.2.23 $\int_1^e \sqrt{1+\frac{1}{x^2}}\,dx$

4.2.24 $\int_{-1}^1 \sqrt{1+\frac{x^2}{1-x^2}}\,dx$

4.2.25 $\int_{-3}^3 \sqrt{1+\frac{x^2}{81-9x^2}}\,dx$

4.2.26 $\int_1^2 \sqrt{1+\frac{1}{x^4}}\,dx$

4.3.1 $8\pi\sqrt{3}-\dfrac{16\pi\sqrt{2}}{3}$

4.3.3 $\dfrac{730\pi\sqrt{730}}{27}-\dfrac{10\pi\sqrt{10}}{27}$

4.3.4 $\pi+2\pi e+\dfrac{1}{4}\pi e^2-\dfrac{\pi}{4e^2}-\dfrac{2\pi}{e}$

4.3.6 $8\pi^2$

4.3.7 $2\pi+\dfrac{8\pi^2}{3\sqrt{3}}$

4.3.8

$$\begin{aligned} a>b &\quad : \quad 2\pi b^2+\frac{2\pi a^2 b}{\sqrt{a^2-b^2}}\arcsin(\sqrt{a^2-b^2}/a), \\ a<b &\quad : \quad 2\pi b^2+\frac{2\pi a^2 b}{\sqrt{b^2-a^2}}\ln\left(\frac{b}{a}+\frac{\sqrt{b^2-a^2}}{a}\right) \end{aligned}$$

4.4.1 $15/2$

4.4.2 5

4.4.3 $16/5$

4.4.5 $\bar{x}=45/28$, $\bar{y}=93/70$

4.4.6 $\bar{x}=0$, $\bar{y}=4/(3\pi)$

4.4.7 $\bar{x} = 1/2, \bar{y} = 2/5$

4.4.8 $\bar{x} = 0, \bar{y} = 8/5$

4.4.9 $\bar{x} = 4/7, \bar{y} = 2/5$

4.4.10 $\bar{x} = \bar{y} = 1/5$

4.4.11 $\bar{x} = 0, \bar{y} = 28/(9\pi)$

4.4.12 $\bar{x} = \bar{y} = 28/(9\pi)$

4.4.13 $\bar{x} = 0, \bar{y} = 244/(27\pi) \approx 2.88$

5.1.2 $y = \arctan t + C$

5.1.3 $y = \dfrac{t^{n+1}}{n+1} + 1$

5.1.4 $y = t\ln t - t + C$

5.1.5 $y = n\pi$, for any integer n.

5.1.6 none

5.1.7 $y = \pm\sqrt{t^2 + C}$

5.1.8 $y = \pm 1, y = (1 + Ae^{2t})/(1 - Ae^{2t})$

5.1.9 $y^4/4 - 5y = t^2/2 + C$

5.1.10 $y = (2t/3)^{3/2}$

5.1.11 $y = M + Ae^{-kt}$

5.1.12 $\dfrac{10\ln(15/2)}{\ln 5} \approx 2.52$ minutes

5.1.13 $y = \dfrac{M}{1 + Ae^{-Mkt}}$

5.1.14 $y = 2e^{3t/2}$

5.1.15 $t = -\dfrac{\ln 2}{k}$

5.1.16 $600e^{-6\ln 2/5} \approx 261$ mg; $\dfrac{5\ln 300}{\ln 2} \approx 41$ days

5.1.17 $100e^{-200\ln 2/191} \approx 48$ mg; $\dfrac{5730\ln 50}{\ln 2} \approx 32339$ years

5.1.18 $y = y_0 e^{t\ln 2}$

5.1.19 $500e^{-5\ln 2/4} \approx 210$ g

5.2.1 $y = Ae^{-5t}$

5.2.2 $y = Ae^{2t}$

5.2.3 $y = Ae^{-\arctan t}$

5.2.4 $y = Ae^{-t^3/3}$

5.2.5 $y = 4e^{-t}$

5.2.6 $y = -2e^{3t-3}$

5.2.7 $y = e^{1+\cos t}$

5.2.8 $y = e^2 e^{-e^t}$

5.2.9 $y = 0$

5.2.10 $y = 0$

5.2.11 $y = 4t^2$

5.2.12 $y = -2e^{(1/t)-1}$

5.2.13 $y = e^{1-t^{-2}}$

5.2.14 $y = 0$

5.2.15 $k = \ln 5$, $y = 100e^{-t\ln 5}$

5.2.16 $k = -12/13$, $y = \exp(-13t^{1/13})$

5.2.17 $y = 10^6 e^{t\ln(3/2)}$

5.2.18 $y = 10e^{-t\ln(2)/6}$

5.3.1 $y = Ae^{-4t} + 2$

5.3.2 $y = Ae^{2t} - 3$

5.3.3 $y = Ae^{-(1/2)t^2} + 5$

5.3.4 $y = Ae^{-e^t} - 2$

5.3.5 $y = Ae^t - t^2 - 2t - 2$

5.3.6 $y = Ae^{-t/2} + t - 2$

5.3.7 $y = At^2 - \dfrac{1}{3t}$

5.3.8 $y = \dfrac{c}{t} + \dfrac{2}{3}\sqrt{t}$

5.3.9 $y = A\cos t + \sin t$

5.3.10 $y = \dfrac{A}{\sec t + \tan t} + 1 - \dfrac{t}{\sec t + \tan t}$

5.4.1 $y(1) \approx 1.355$

5.4.2 $y(1) \approx 40.31$

5.4.3 $y(1) \approx 1.05$

5.4.4 $y(1) \approx 2.30$

6.1.1 1

6.1.3 0

6.1.4 1

6.1.5 1

6.1.6 0

6.2.1 $\lim\limits_{n\to\infty} n^2/(2n^2+1) = 1/2$

6.2.2 $\lim\limits_{n\to\infty} 5/(2^{1/n}+14) = 1/3$

6.2.3 $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so $\displaystyle\sum_{n=1}^{\infty} 3\frac{1}{n}$ diverges

6.2.4 $-3/2$

6.2.5 11

6.2.6 20

6.2.7 $3/4$

6.2.8 $3/2$

6.2.9 $3/10$

6.3.1 diverges

6.3.2 diverges

6.3.3 converges

6.3.4 converges

6.3.5 converges

6.3.6 converges

6.3.7 diverges

6.3.8 converges

6.3.9 $N = 5$

6.3.10 $N = 10$

6.3.11 $N = 1687$

6.3.12 any integer greater than e^{200}

6.4.1 converges

6.4.2 converges

6.4.3 diverges

6.4.4 converges

6.4.5 0.90

6.4.6 0.95

6.5.1 converges

6.5.2 converges

6.5.3 converges

6.5.4 diverges

6.5.5 diverges

6.5.6 diverges

6.5.7 converges

6.5.8 diverges

6.5.9 converges

6.5.10 diverges

6.6.1 converges absolutely

6.6.2 diverges

6.6.3 converges conditionally

6.6.4 converges absolutely

6.6.5 converges conditionally

6.6.6 converges absolutely

6.6.7 diverges

6.6.8 converges conditionally

6.7.5 (a) converges

(b) converges

(c) converges

(d) diverges

6.8.1 (a) $R = 1, I = (-1, 1)$

(b) $R = \infty, I = (-\infty, \infty)$

(c) $R = e, I = (2 - e, 2 + e)$

(d) $R = 0$, converges only when $x = 2$

(e) $R = 1, I = [-6, -4]$

6.8.2 $R = e$

6.9.1 the alternating harmonic series

6.9.2 $\sum_{n=0}^{\infty}(n+1)x^n$

6.9.3 $\sum_{n=0}^{\infty}(n+1)(n+2)x^n$

6.9.4 $\sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2}x^n$, $R = 1$

6.9.5 $C+\sum_{n=0}^{\infty}\frac{-1}{(n+1)(n+2)}x^{n+2}$

6.10.1 (a) $\sum_{n=0}^{\infty}(-1)^n x^{2n}/(2n)!$, $R = \infty$

(b) $\sum_{n=0}^{\infty}x^n/n!$, $R = \infty$

(c) $\sum_{n=0}^{\infty}(-1)^n\frac{(x-5)^n}{5^{n+1}}$, $R = 5$

(d) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-1)^n}{n}$, $R = 1$

(e) $\ln(2)+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-2)^n}{n2^n}$, $R = 2$

(f) $\sum_{n=0}^{\infty}(-1)^n(n+1)(x-1)^n$, $R = 1$

(g) $1+\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!2^n}x^n = 1+\sum_{n=1}^{\infty}\frac{(2n-1)!}{2^{2n-1}(n-1)!\,n!}x^n$, $R = 1$

(h) $x+x^3/3$

(i) $\sum_{n=0}^{\infty}(-1)^n x^{4n+1}/(2n)!$

(j) $\sum_{n=0}^{\infty}(-1)^n x^{n+1}/n!$

6.11.1 $1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\cdots+\frac{x^{12}}{12!}$

6.11.2 1000; 8

6.11.3 $x+\frac{x^3}{3}+\frac{2x^5}{15}$, error ± 1.27.

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